5.3 Ruled surfaces and surfaces of revolution

Level surfaces have an ‘algebraic’ origin, in that they arise from a function $f(x, y, z)$. On the other hand, the two classes of surfaces considered in this section arise from geometric constructions.

Example 5.3.1

A ruled surface is a surface that is a union of straight lines, called the rulings (or sometimes the generators) of the surface.
Suppose that $C$ is a curve in $\mathbb{R}^3$ that meets each of these lines. Any point $p$ of the surface lies on one of the given straight lines, which intersects $C$ at $q$, say. If $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^3$ is a parametrization of $C$ with $\gamma(u) = q$, and if $\delta(u)$ is a non-zero vector in the direction of the line passing through $\gamma(u)$, then

$$p = \gamma(u) + v\delta(u),$$

for some scalar $v$. Denoting the right-hand side by $\sigma(u, v)$, it is clear that $\sigma : U \rightarrow \mathbb{R}^3$ is a smooth map, where $U = \{(u, v) \in \mathbb{R}^2 \mid \alpha < u < \beta\}$. Moreover, denoting $d/du$ by a dot,

$$\sigma_u = \dot{\gamma} + v\dot{\delta}, \quad \sigma_v = \delta.$$

Thus, $\sigma$ is regular if $\dot{\gamma} + v\dot{\delta}$ and $\delta$ are linearly independent. This will be true, for example, if $\dot{\gamma}$ and $\delta$ are linearly independent and $v$ is sufficiently small. Thus, to get a surface, the curve $C$ must never be tangent to the rulings.

An important special case is that in which the rulings are all parallel to each other; the ruled surface $S$ is then called a generalized cylinder. In the above notation, we can take $\delta$ to be a constant unit vector, say $a$, parallel to the rulings, and the parametrization becomes

$$\sigma(u, v) = \gamma(u) + va.$$

Since

$$\sigma(u, v) = \sigma(u', v') \iff \gamma(u) - \gamma(u') = (v' - v)a,$$

for $\sigma$ to be a injective (and hence a surface patch), no straight line parallel to $a$ should meet $\gamma$ in more than one point. Finally, $\sigma_u = \dot{\gamma}$, $\sigma_v = a$, so $\sigma$ is regular if and only if $\gamma$ is never tangent to the rulings.

The parametrization is simplest when $\gamma$ lies in a plane perpendicular to $a$ (in fact, this can always be achieved by replacing $\gamma$ by its perpendicular projection onto such a plane – see Exercise 5.3.3). The regularity condition is then
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clearly satisfied provided \( \dot{\gamma} \) is never zero, i.e., provided \( \gamma \) is regular. We might as well take the plane to be the \( xy \)-plane and \( a = (0, 0, 1) \) to be parallel to the \( z \)-axis. Then, \( \gamma(u) = (f(u), g(u), 0) \) for some smooth functions \( f \) and \( g \), and the parametrization becomes

\[
\sigma(u, v) = (f(u), g(u), v).
\]

For example, starting with a circle, we get a circular cylinder. Taking the circle to have centre the origin, radius 1 and to lie in the \( xy \)-plane, it can be parametrized by

\[
\gamma(u) = (\cos u, \sin u, 0),
\]
defined for \( 0 < u < 2\pi \) and \( -\pi < u < \pi \), say. This gives the atlas for the unit cylinder found in Example 4.1.3.

The second special case we shall consider is that in which the rulings all pass through a certain fixed point, say \( v \); then \( \mathcal{S} \) is called a generalized cone with vertex \( v \).

We can take \( \delta(u) = \gamma(u) - v \), giving

\[
\sigma(u, v) = (1 + v)\gamma(u) - v\cdot v.
\]

Now,

\[
\sigma(u, v) = \sigma(u', v') \iff (1 + v)\gamma(u) - (1 + v')\gamma(u') + (v' - v)v = 0;
\]
since \( (1 + v) - (1 + v') + (v' - v) = 0 \), the equation on the right-hand side means that the points \( v, \gamma(u) \) and \( \gamma(u') \) are collinear. So, for \( \sigma \) to be a surface patch,
no straight line passing through \( v \) should pass through more than one point of \( \gamma \) (in particular, \( \gamma \) should not pass through \( v \)). Finally, we have \( \sigma_u = (1 + v)\dot{\gamma} \),
\( \sigma_v = \gamma - v \), so \( \sigma \) is regular provided \( v \neq -1 \), i.e., the vertex of the cone is omitted (cf. Example 4.1.5), and none of the straight lines forming the cone is tangent to \( \gamma \).

The parametrization is simplest when \( \gamma \) lies in a plane. If this plane contains \( v \), the cone is simply part of that plane. Otherwise, we can take \( v \) to be the origin and the plane to be \( z = 1 \). Then, \( \gamma(u) = (f(u), g(u), 1) \) for some smooth functions \( f \) and \( g \), and the parametrization takes the form
\[
\sigma(u, v) = v(f(u), g(u), 1),
\]
after making the reparametrization \( v \mapsto v - 1 \).

**Example 5.3.2**

A *surface of revolution* is the surface obtained by rotating a plane curve, called the *profile curve*, around a straight line in the plane. The circles obtained by rotating a fixed point on the profile curve around the axis of rotation are called the *parabols* of the surface, and the curves on the surface obtained by rotating the profile curve through a fixed angle are called its *meridians*. (This agrees with the use of these terms in geography, if we think of the earth as the surface obtained by rotating a circle passing through the poles about the polar axis and we take \( u \) and \( v \) to be latitude and longitude, respectively.)
Let us take the axis of rotation to be the $z$-axis and the plane to be the $xz$-plane. Any point $p$ of the surface is obtained by rotating some point $q$ of the profile curve through an angle $v$ (say) around the $z$-axis. If

$$\gamma(u) = (f(u), 0, g(u))$$

is a parametrization of the profile curve containing $q$, $p$ is of the form

$$\mathbf{\sigma}(u, v) = (f(u) \cos v, f(u) \sin v, g(u)).$$

To check regularity, we compute (with a dot denoting $d/du$):

$$\mathbf{\sigma}_u = (\dot{f} \cos v, \dot{f} \sin v, \dot{g}), \quad \mathbf{\sigma}_v = (-f \sin v, f \cos v, 0),$$

$$\therefore \mathbf{\sigma}_u \times \mathbf{\sigma}_v = (\dot{f} \cos v, -f \dot{g} \sin v, f \dot{f}),$$

$$\therefore \|\mathbf{\sigma}_u \times \mathbf{\sigma}_v\|^2 = f^2(\dot{f}^2 + \dot{g}^2).$$

Thus, $\mathbf{\sigma}_u \times \mathbf{\sigma}_v$ will be non-vanishing if $f(u)$ is never zero, i.e., if $\gamma$ does not intersect the $z$-axis, and if $\dot{f}$ and $\dot{g}$ are never zero simultaneously, i.e., if $\gamma$ is regular. In this case, we might as well assume that $f(u) > 0$, so that $f(u)$ is the distance of $\mathbf{\sigma}(u, v)$ from the axis of rotation. Then, $\mathbf{\sigma}$ is injective provided that $\gamma$ does not self-intersect and the angle of rotation $v$ is restricted to lie in an open interval of length $\leq 2\pi$. Under these conditions, surface patches of the form $\mathbf{\sigma}$ give the surface of revolution the structure of a surface.

**EXERCISES**

5.3.1 The surface obtained by rotating the curve $x = \cosh z$ in the $xz$-plane around the $z$-axis is called a **catenoid**. Describe an atlas for this surface.
5.3.2 Show that
\[ \sigma(u, v) = (\sech u \cos v, \sech u \sin v, \tanh u) \]
is a regular surface patch for \( S^2 \) (it is called Mercator’s projection). Show that meridians and parallels on \( S^2 \) correspond under \( \sigma \) to perpendicular straight lines in the plane. (This patch is ‘derived’ in Exercise 6.3.3.)

5.3.3 Show that, if \( \sigma(u, v) \) is the (generalized) cylinder in Example 5.3.1:

(i) The curve \( \tilde{\gamma}(u) = \gamma(u) - (\gamma(u) \cdot a)a \) is contained in a plane perpendicular to \( a \).

(ii) \( \sigma(u, v) = \tilde{\gamma}(u) + \tilde{v}a \), where \( \tilde{v} = v + \gamma(u) \cdot a \).

(iii) \( \tilde{\sigma}(u, \tilde{v}) = \tilde{\gamma}(u) + \tilde{v}a \) is a reparametrization of \( \sigma(u, v) \).

This exercise shows that, when considering a generalized cylinder \( \sigma(u, v) = \gamma(u) + va \), we can always assume that the curve \( \gamma \) is contained in a plane perpendicular to the vector \( a \).

5.3.4 Consider the ruled surface
\[ \sigma(u, v) = \gamma(u) + v\delta(u), \] (5.5)
where \( \| \delta(u) \| = 1 \) and \( \dot{\delta}(u) \neq 0 \) for all values of \( u \) (a dot denotes \( d/du \)). Show that there is a unique point \( \Gamma(u) \), say, on the ruling through \( \gamma(u) \) at which \( \dot{\delta}(u) \) is perpendicular to the surface. The curve \( \Gamma \) is called the line of striction of the ruled surface \( \sigma \) (of course, it need not be a straight line). Show that \( \dot{\Gamma} \cdot \dot{\delta} = 0 \). Let \( \tilde{v} = v + \frac{\gamma \cdot \dot{\delta}}{\| \delta \|} \), and let \( \tilde{\sigma}(u, \tilde{v}) \) be the corresponding reparametrization of \( \sigma \). Then, \( \tilde{\sigma}(u, \tilde{v}) = \Gamma(u) + \tilde{v}\delta(u) \). This means that, when considering ruled surfaces as in (5.5), we can always assume that \( \dot{\gamma} \cdot \dot{\delta} = 0 \). We shall make use of this in Chapter 12.