Weekplan: Priority Queues and Heaps

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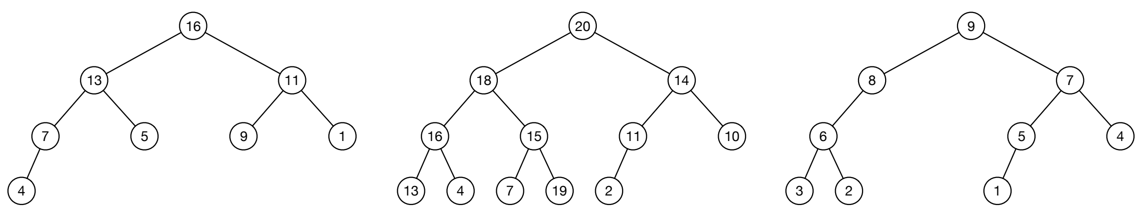
Reading

Introduction to Algorithms, Cormen, Rivest, Leisersons and Stein (CLRS): Chapter 6 + Appendix B.5

Exercises

1 Heap Properties and Run by Hand Solve the following exercises.

1.1 [w] Which of the following trees are heaps?



(a) (b) (c)

1.2 [w] Which of the following arrays are heaps? Index 0 is not used and is therefore marked with —

$$A = [-, 9, 7, 8, 3, 4] \qquad B = [-, 12, 4, 7, 1, 2, 10] \qquad C = [-, 5, 7, 8, 3]$$

1.3 [w] Let $S = 4, 8, 11, 5, 21, \star, 2, \star$ be a sequence of operations where a number corresponds to an insertion of that number and $\star$ corresponds to an ExtractMax operation. Starting with an empty heap H, show how H looks after each operation in S.

1.4 Is a sorted array a heap?

1.5 Where can the minimum element be found in a (max-)heap?

1.6 [BSc] Show that Insert, ExtractMax and IncreaseKey maintains the heap property.

1.7 [$*$] CLRS 6.5-9.

2 [w] **Priority Politics** The Kakistocradic Party want you to help them implement their "Fresh Air"-policy. Design a registry of all citizens and their income such that one can efficiently find those with the lowest income and deport them. Specifically, the system must support the following operations.

- Insert(c, i): insert a person with social number c and a yearly income i in the system.
- DeportLowestIncome(): Remove and return the person with the lowest income.

Design an efficient solution for the system.

3 Priority Queue Operations We now want to extend the set of operations on priority queues. We are interested in the following operations.

- REMOVELARGEST(m): remove the m largest elements in the priority queue.
- DELETE(x): remove the element x from the priority queue.
- FUSION(x, y): remove x and y from the priority queue and add the element z with key $x.key + y.key$.
- FINDLARGEST(x): return the elements in the priority queue with key $\geq x$.
- EXTRACTMIN: Remove and return the element with the lowest key.

We want to support these operations efficiently, while maintaining the complexities of the of standard operations. Let n be the number of elements in the priority queue. Solve the following exercises.

3.1 Extend the priority queue to support REMOVELARGEST(m) in $O(m \log n)$ time.

3.2 Extend the priority queue to support DELETE and FUSION in $O(\log n)$ time.

3.3 [$*$] Extend the priority queue to support FINDLARGEST in $O(m)$ time, where m is the number of elements with key $\geq x$.

3.4 [$*$] Extend the priority queue to support EXTRACTMIN in $O(\log n)$ time.

4 Satellite Data Let $A[0..n]$ be a heap represented as an array. Each element x in the heap is represented by an index i og the key stored in $A[i]$. It is often useful to store some extra information (called *satellite data*) associated with an element (for instance if we want to store persons in a heap the satellite data could be age, gender, heigh, weight, etc). Show how to support access to satellite data in $O(1)$ time only given the index i while still maintaining the running times for the standard heap operations.

5 Heap Properties Let T be a complete binary tree of height h. Solve the following exercises.

5.1 Show the number of nodes in T is $n = 2^{h+1} - 1$. *Hint:* we know $n = 1 + 2 + 4 + \cdots 2^h$. Multiply the sum by 2 and subtract the sum.

5.2 [BSc] Show that the sum, $S = n/4 \cdot 1 + n/8 \cdot 2 + n/16 \cdot 3 + n/32 \cdot 4 + \cdots = \Theta(n)$. *Hint:* Calculate $S - S/2$.

6 Implementation of Heaps We are interested in implementing a priority queue using a heap represented by an array. Solve the following exercises.

6.1 [†] Implement the INSERT and EXTRACTMAX operations.

7 Sums Let $A[0..n-1]$ be an array of integers. We are interested in the following operations on A.

- SUM(i, j): compute $A[i] + A[i+1] + \cdots + A[j]$.
- CHANGE(i, x): set $A[i] = x$.

Solve the following exercises.

7.1 [w] Give a data structure that supports SUM in $O(1)$ time and uses $O(n^2)$ space.

7.2 [$*$] Give a data structure that supports SUM in $O(1)$ time and uses $O(n)$ space.

7.3 [$**$] Give a data structure that supports both SUM and CHANGE in $O(\log n)$ time and uses $O(n)$ space.

M Mandatory Exercise: Heaps and Arrays Let $A[0..n]$ be an array, $n \geq 1$. Solve the following exercises.

- Give an algorithm that decides whether or not A represents a heap.
- Write the pseudo-code for your algorithm.