Weekplan: Introduction to Graphs

The 02105+02326 DTU Algorithms Team

Reading

Introduction to Algorithms, Cormen, Rivest, Leisersons and Stein (CLRS): Introduction to Part VI + Chapter 22.1-22.4 + Appendix B.4-B.5.

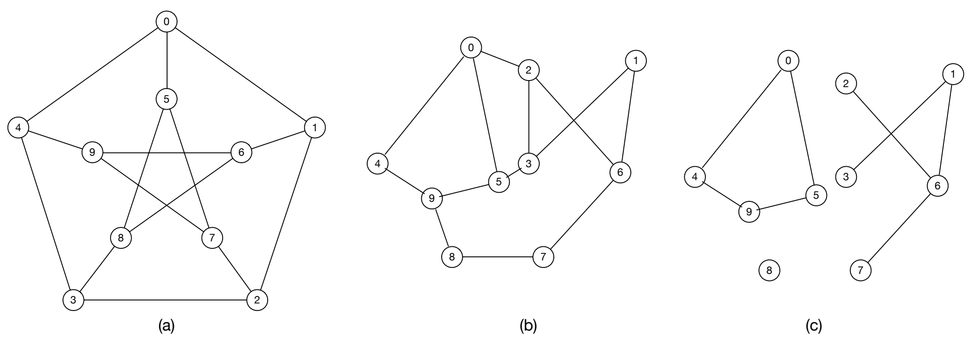


Figure 1: Graphs for the exercises. (a) is the *Petersen graph*.

Exercises

1 Representation, Properties and Algorithms Consider the graphs in Figure 1. Solve the following exercises.

1.1 [w] Show adjacency lists and adjacency matrices for (a) and (c).

1.2 [w] Simulate DFS on (a) starting in vertex 0. Assume the adjacency lists are sorted. Specify the DFS-tree, and discovery and finish times.

1.3 [w] Simulate BFS on (a) starting in vertex 0. Assume the adjacency lists are sorted. Specify the BFS-tree, and the distance for each vertex.

1.4 Specify the connected components of (a), (b), and (c).

1.5 Which of (a), (b), and (c) are bipartite?

2 Depth-First Search using a Stack Explain how to implement DFS without using recursion. *Hint:* use an (explicit) stack.

3 Find a Cycle Give an algorithm that determines if a graph is *cyclic*, ie. contains a cycle. How fast is your algorithm?

4 Number of Shortest Paths Give an algorithm that given two vertices s and t in G returns the *number* of shortest paths between s and t in G.

5 Mazes and Grid Graphs (exam 2010) A $k \times k$ *grid graph* is a graph where the vertices are arranged in k rows each containing k vertices. Only vertices that are adjacent in the horizontal or vertical direction may have an edge between them. See Figure 2(a). Solve the following exercises.

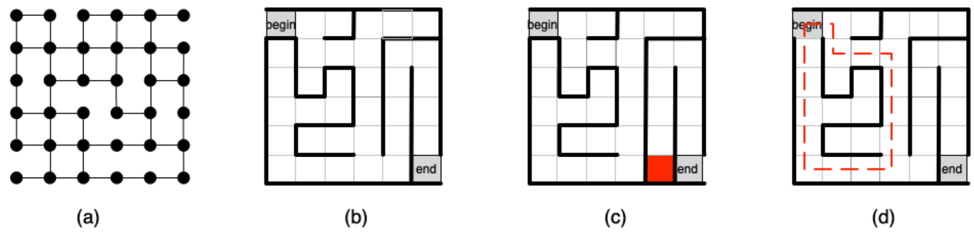


Figure 2: (a) a 6×6 grid graph. (b), (c), and (d) are 6×6 mazes. (b) is happy, (c) is unhappy since the end field cannot be reached, and (d) is unhappy since it contains a circular path.

5.1 Let the n and m denote the number of vertices and edges, respectively, in a $k \times k$ grid graph. Express n and m as a function of k in asymptotic notation.

A $k \times k$ *maze* is a square drawing consisting k^2 *fields* arranged in k rows each containing k fields. Each of the four sides of each field is either a *wall* or *empty*. A *walk* in a maze is a sequence of fields $f_1, \ldots, f_j$ such that any pair f, f' of consecutive fields in the sequence are adjacent in the horizontal or vertical direction and the shared side of f and f' is empty. A special field in the maze is designated as *begin* and another special field is designated as *end*. A maze is *happy* if the following conditions hold:

- There is exactly one unique walk in the maze from begin to end.
- There is a walk from start to any field in the maze.
- There are no circular walks, i.e., walks that start and end in the same field.

A maze that is not happy is *unhappy*. See Figure 2(b)-(d).

5.2 Explain how to model a $k \times k$ maze as a $k \times k$ grid graph.

5.3 Draw the maze in Figure2(b) as a grid graph.

5.4 Give an algorithm, that given a $k \times k$ maze modelled as a $k \times k$ grid graph, determines if the maze is happy. Argue the correctness of your algorithm and analyze it's running time as a function of k.

6 Implementation of Graphs We want to support the following operations on a dynamic graph G.

- ADDEDGE(u, v): add an edge between the vertices u and v.
- ADJACENT(u, v): return if u and v are adjacent in G.
- NEIGHBOURS(v): prints all neighbors of vertex v.

Solve the following exercises.

6.1 [†] Implement the operations on an adjacency matrix.

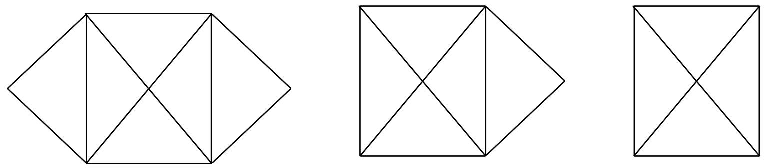
6.2 [†] Implement the operations on an adjacency list.

7 Euler Tours and Euler Paths Let G be a connected graph with n vertices and m edges. An *Euler tour* in G is a cycle that contains all edges in G exactly once. An *Euler path* in G is a path that contains all edges in G exactly once. Solve the following exercises.

7.1 [$*$] Show that G has an Euler tour if and only if all vertices have even degree.

7.2 [$*$] Show that G has an Euler path if and only if 2 or 0 vertices have odd degree.

7.3 Which of the drawings below can you draw without lifting the pencil? Can you start and end at the same place?



7.4 Give an $O(n+m)$ time algorithm that determines if G has an Euler tour.

7.5 [$*$] Give an $O(n+m)$ algorithm that finds an Euler tour in G if it exists.

8 Diameter of Trees Let T be a tree with n vertices. The *diameter* of T is the longest shortest path between a pair of vertices in T. Solve the following exercises.

8.1 Give algorithm to compute the diameter of T in $O(n^2)$ time.

8.2 [$**$] Give algorithm to compute the diameter of T in $O(n)$ time.