Shortest Paths

- Shortest Paths
- Properties of Shortest Paths
- Dijkstra's Algorithm
- Shortest Paths on DAGs

Philip Bille

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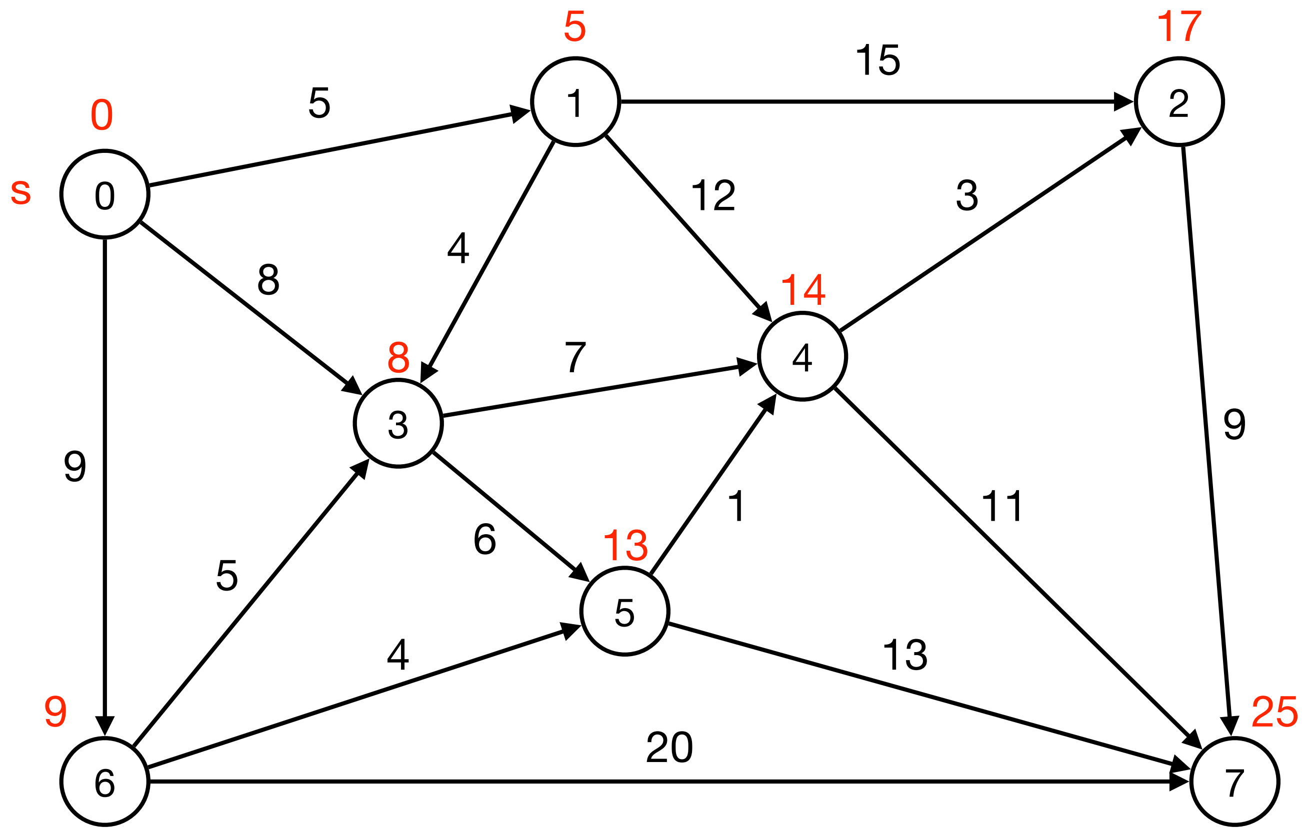
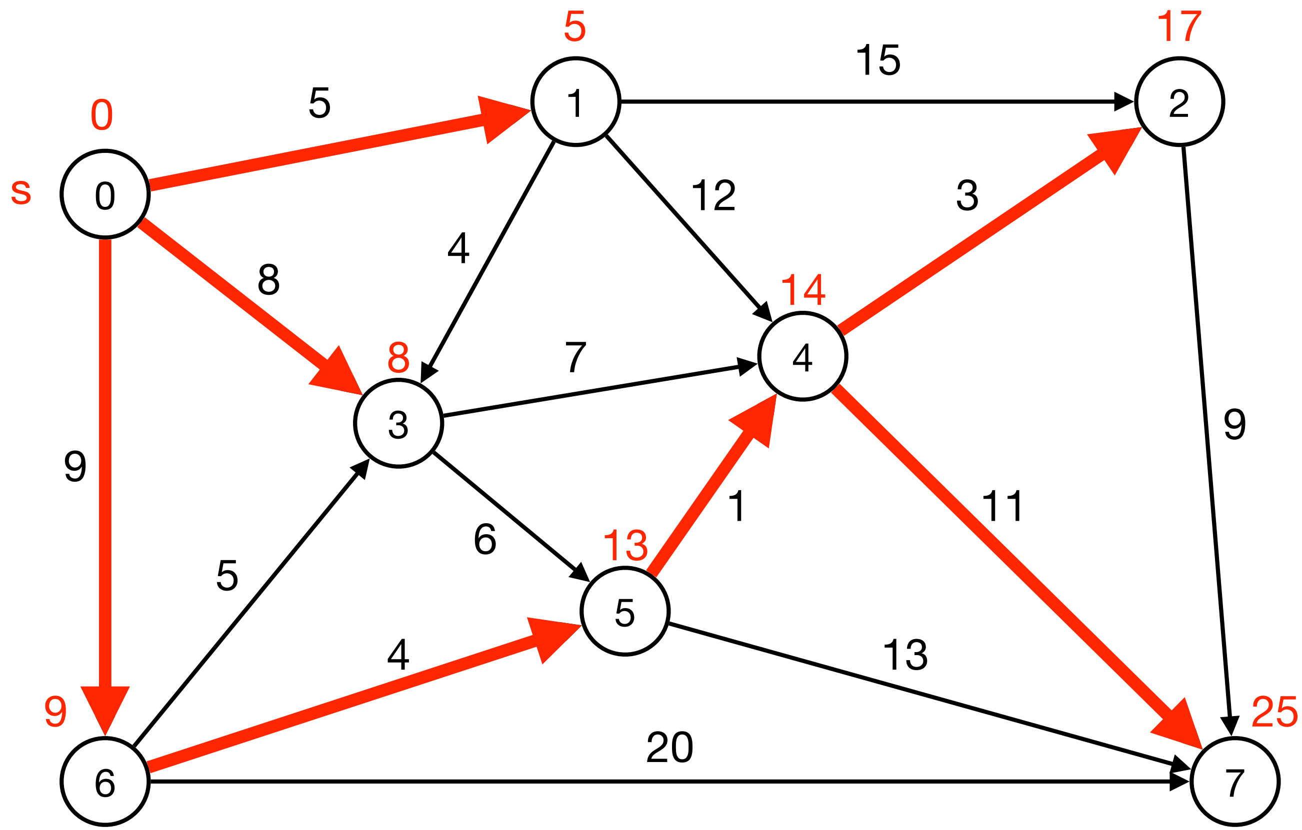
Shortest Paths

- Shortest paths. Given a directed, weighted graph G and vertex s, find shortest path from s to all vertices in G.



Shortest Paths

- Shortest paths. Given a directed, weighted graph G and vertex s, find shortest path from s to all vertices in G.
- Shortest path tree. Represent shortest paths in a tree from s.



Applications

- Routing, scheduling, pipelining, ...

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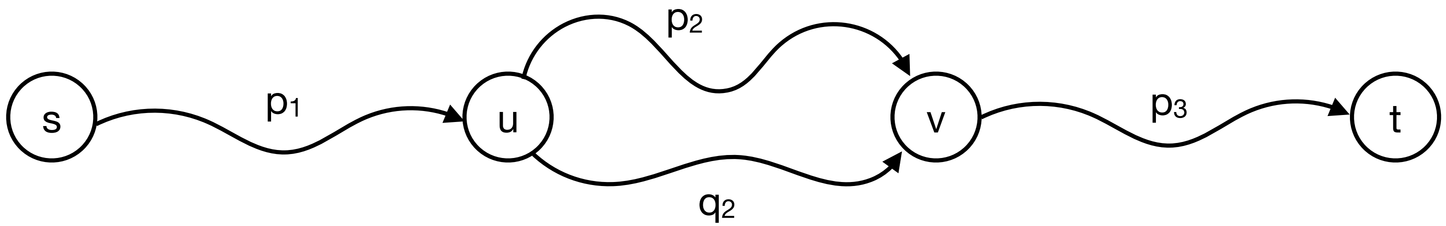
Properties of Shortest Paths

- Assume for simplicity:
    - All vertices are reachable from s.
- $\Longrightarrow$ a (shortest) path to each vertex always exists.

Properties of Shortest Paths

- Subpath property. Any subpath of a shortest path is a shortest path.
- Proof.
  - Consider shortest path from s to t consisting of p_1, p_2 and p_3.



  - Assume q_2 is shorter than p_2.
  - $\Longrightarrow$ Then p_1, q_2 and p_3 is shorter than p.

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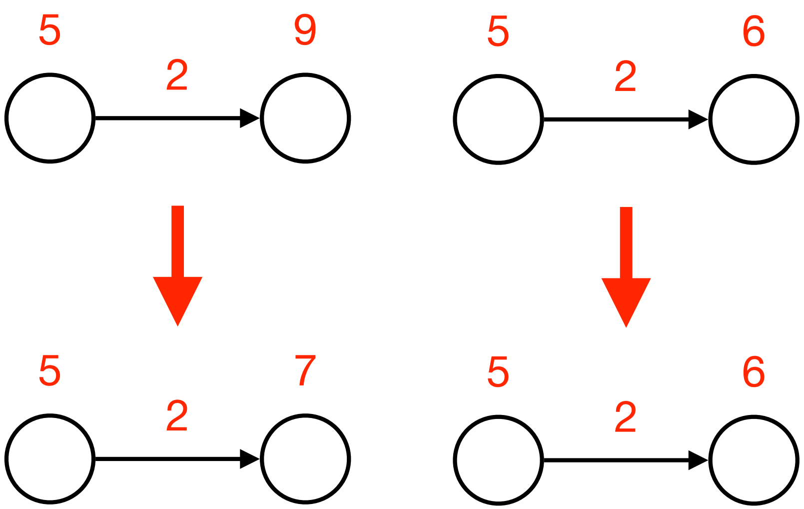
Dijkstra's Algorithm

- Goal. Given a directed, weighted graph with non-negative weights and a vertex s, compute shortest paths from s to all vertices.

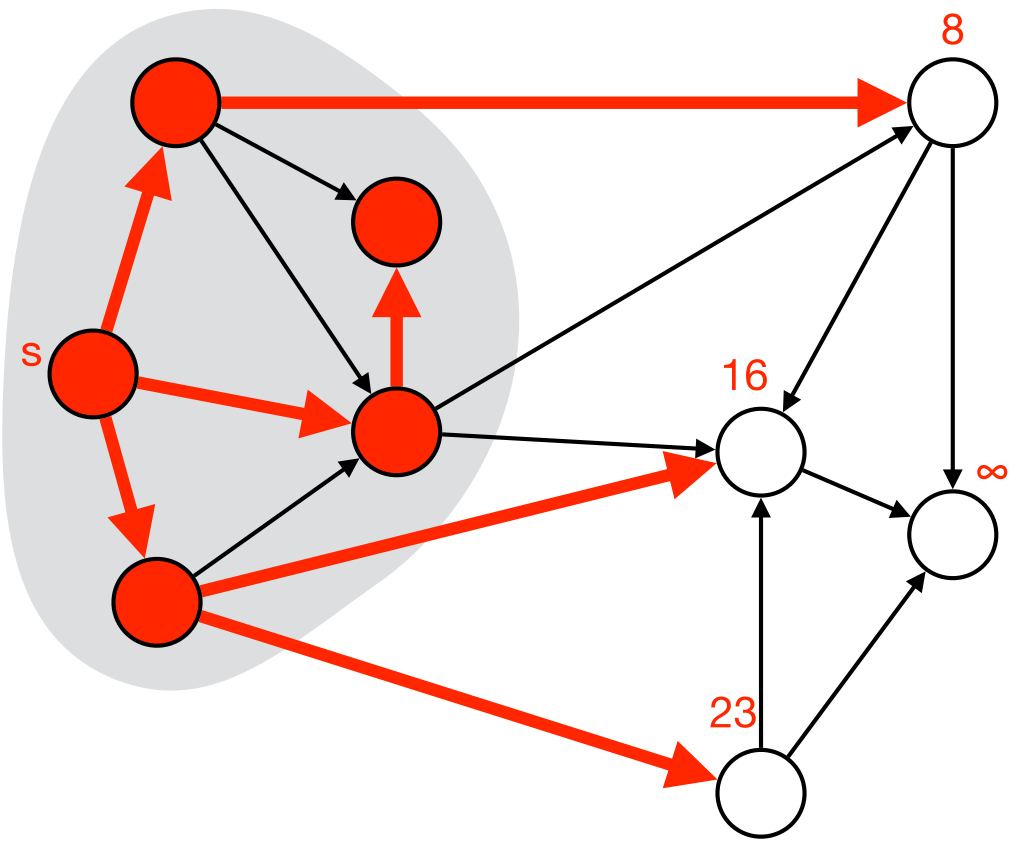
- Dijkstra's algorithm.
  - Maintains distance estimate v.d for each vertex v = length of shortest known path from s to v.
  - Updates distance estimates by relaxing edges.

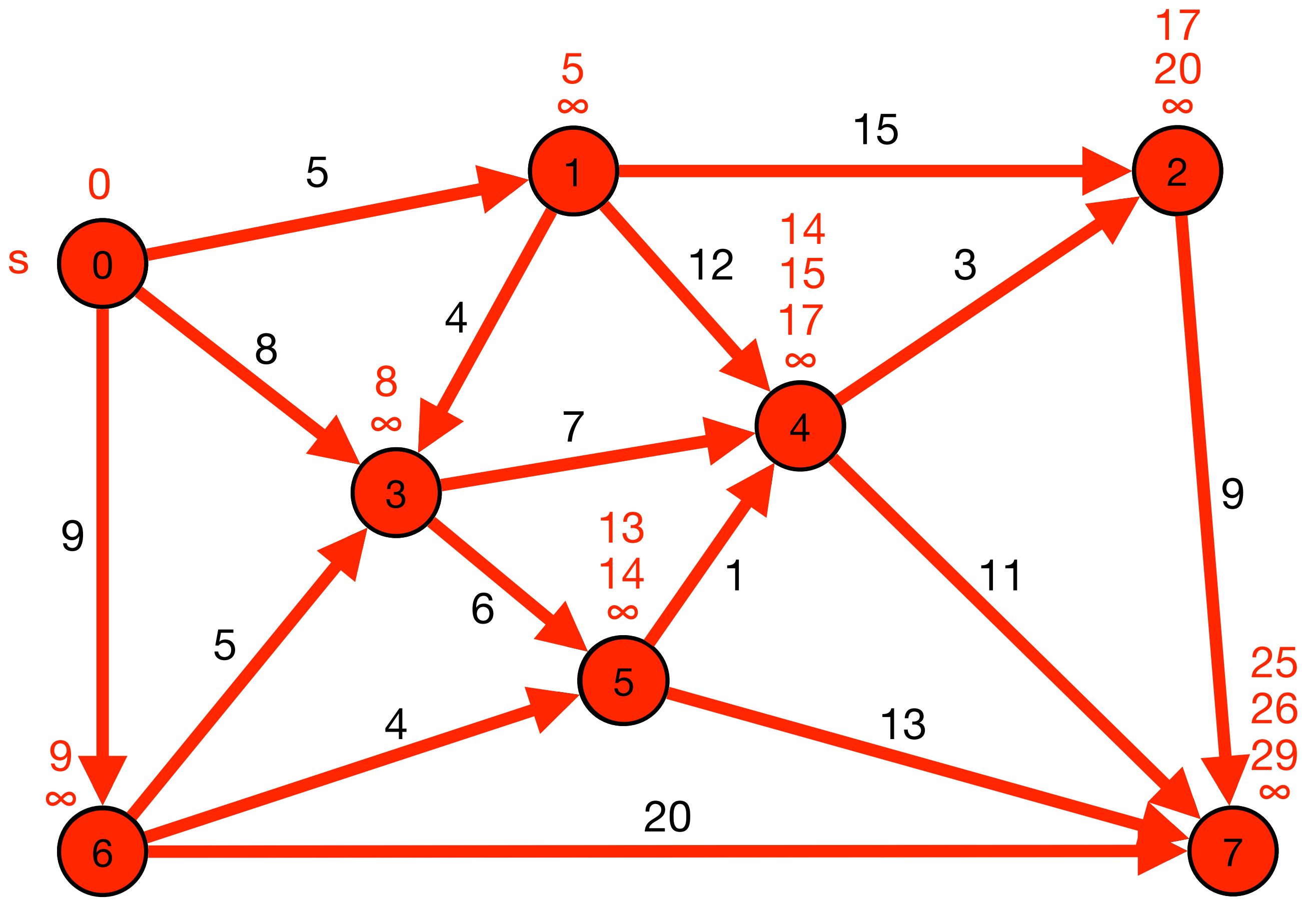



```
RELAX(u,v)
    if (v.d > u.d + w(u,v))
        v.d = u.d + w(u,v)
```

Dijkstra's Algorithm

- Initialize s.d = 0 and v.d = ∞ for all vertices v ∈ V\{s}.
- Grow tree T from s.
- In each step, add vertex with smallest distance estimate to T.
- Relax all outgoing edges of v.





s
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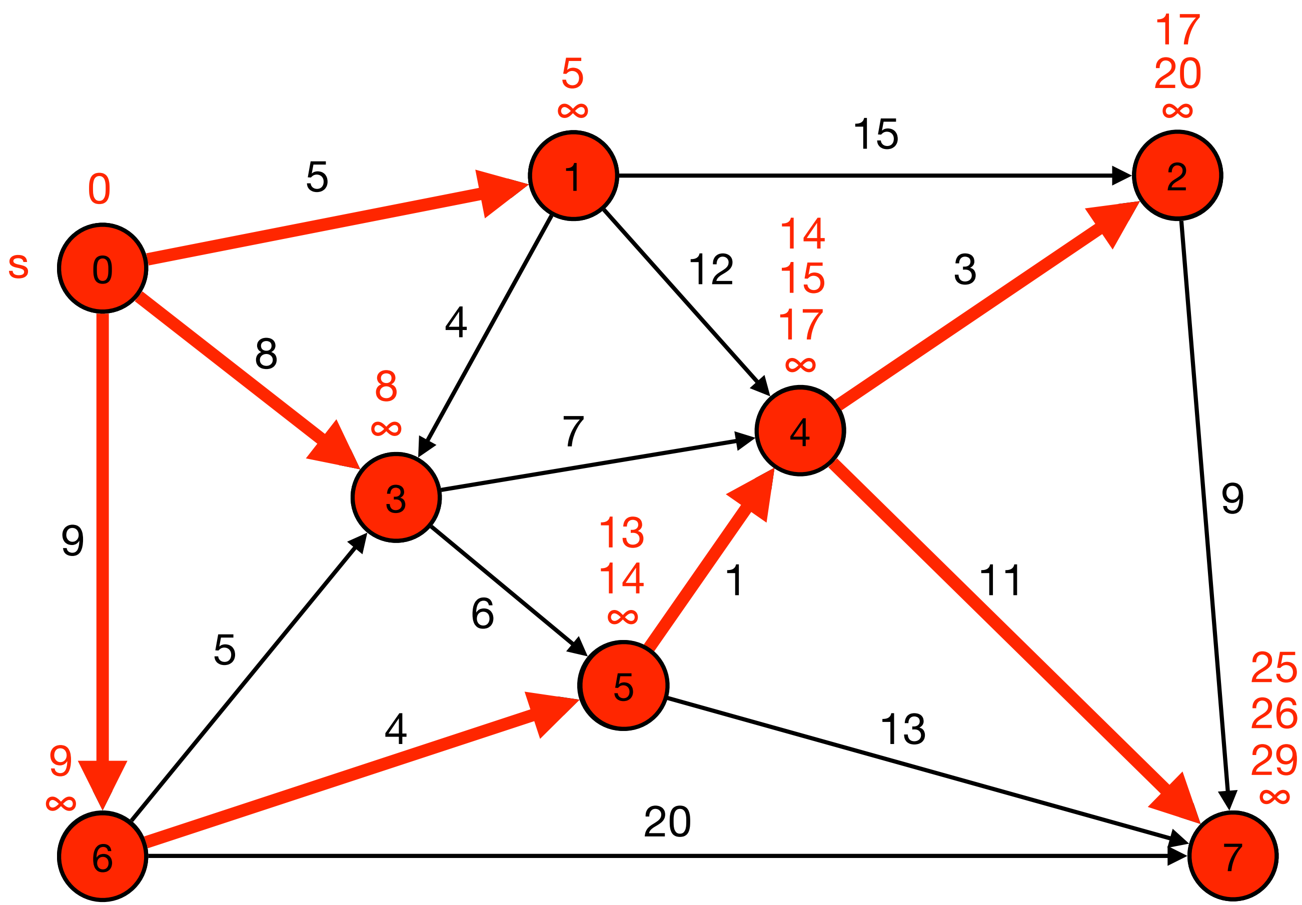

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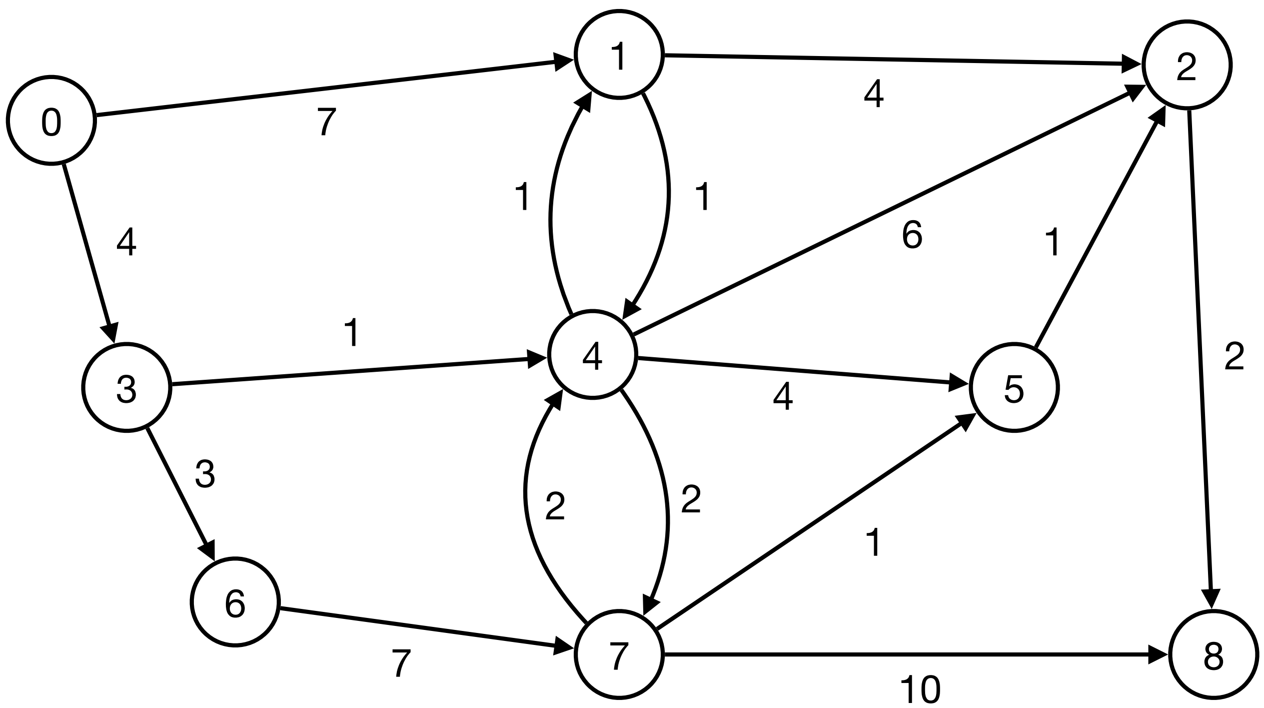
7

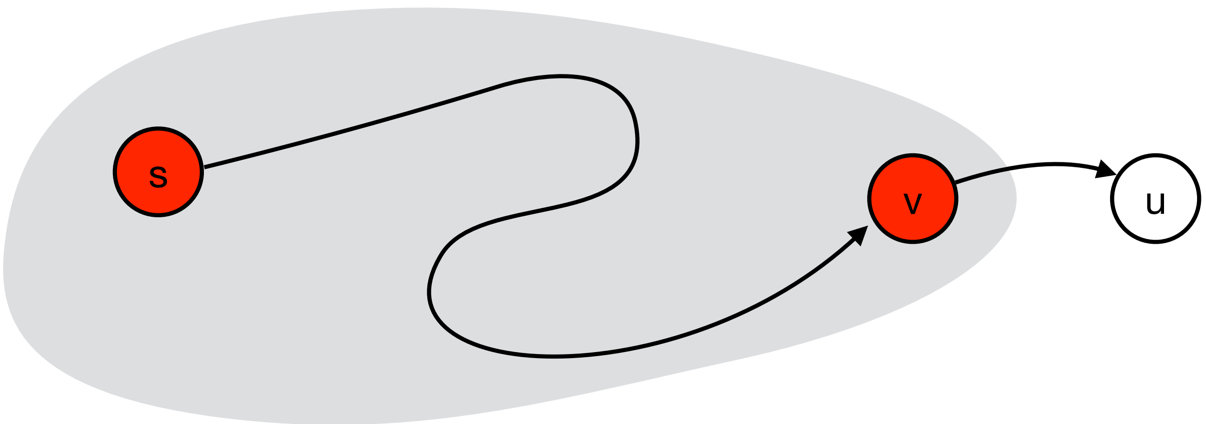
Dijkstra's Algorithm

- Initialize s.d = 0 and v.d = ∞ for all vertices v ∈ V\{s}.
- Grow tree T from s.
- In each step, add vertex with smallest distance estimate to T.
- Relax all outgoing edges of v.
- Exercise. Show execution of Dijkstra's algorithm from vertex 0.



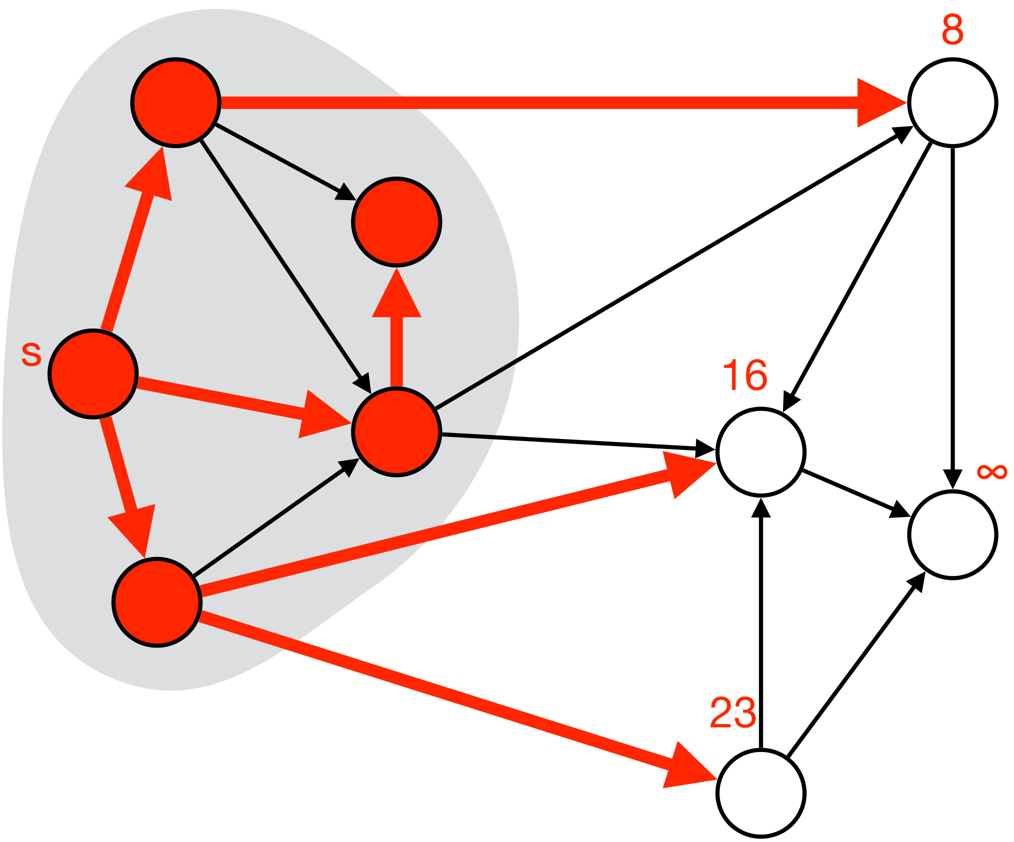
Dijkstra's Algorithm

- Lemma. Dijkstra's algorithms computes shortest paths.
- Proof.
    - Consider some step after growing tree T and assume distances in T are correct.
    - Consider closest vertex u of s not in T.
    - Shortest path from s to u ends with an edge e = (v,u).
    - v is closer than u to s $\Longrightarrow$ v is in T. (u was closest not in T)
    - $\Longrightarrow$ shortest path to u is in T except e.
    - e is relaxed $\Longrightarrow$ distance estimate to v is correct shortest distance.
    - Dijkstra adds e to T $\Longrightarrow$ T is shortest path tree after n-1 steps.



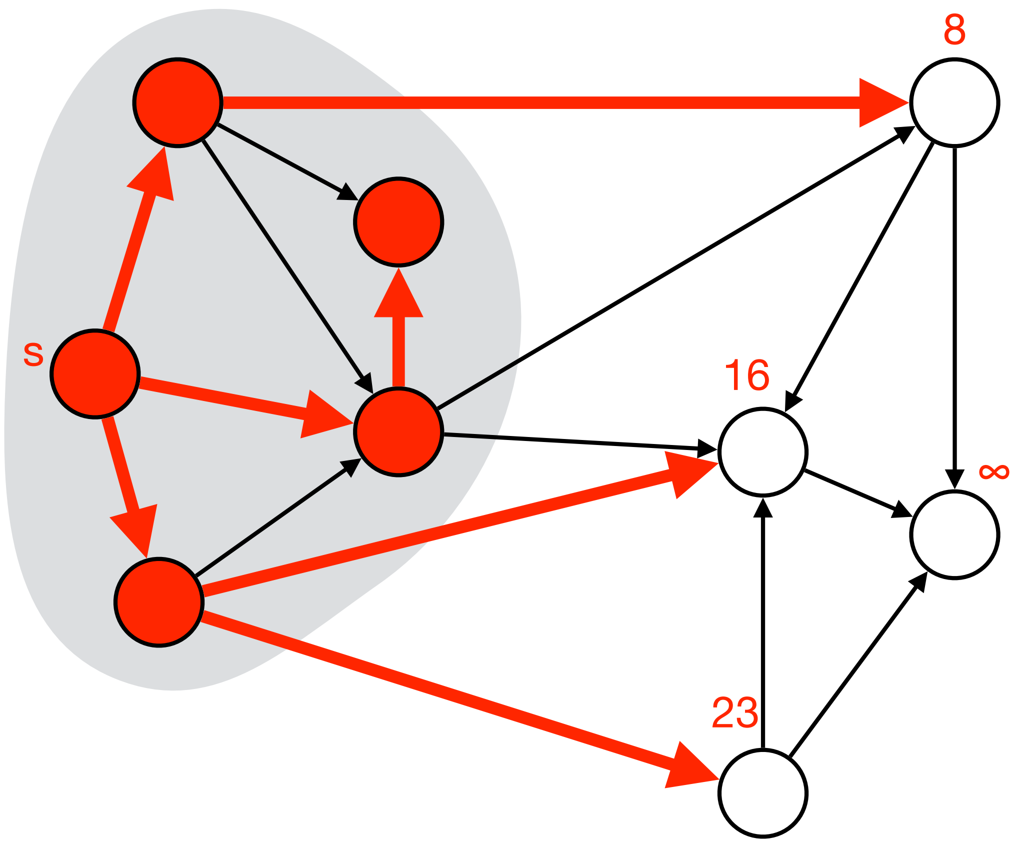
Dijkstra's Algorithm

- Implementation. How do we implement Dijkstra's algorithm?
- Challenge. Find vertex with smallest distance estimate.



Dijkstra's Algorithm

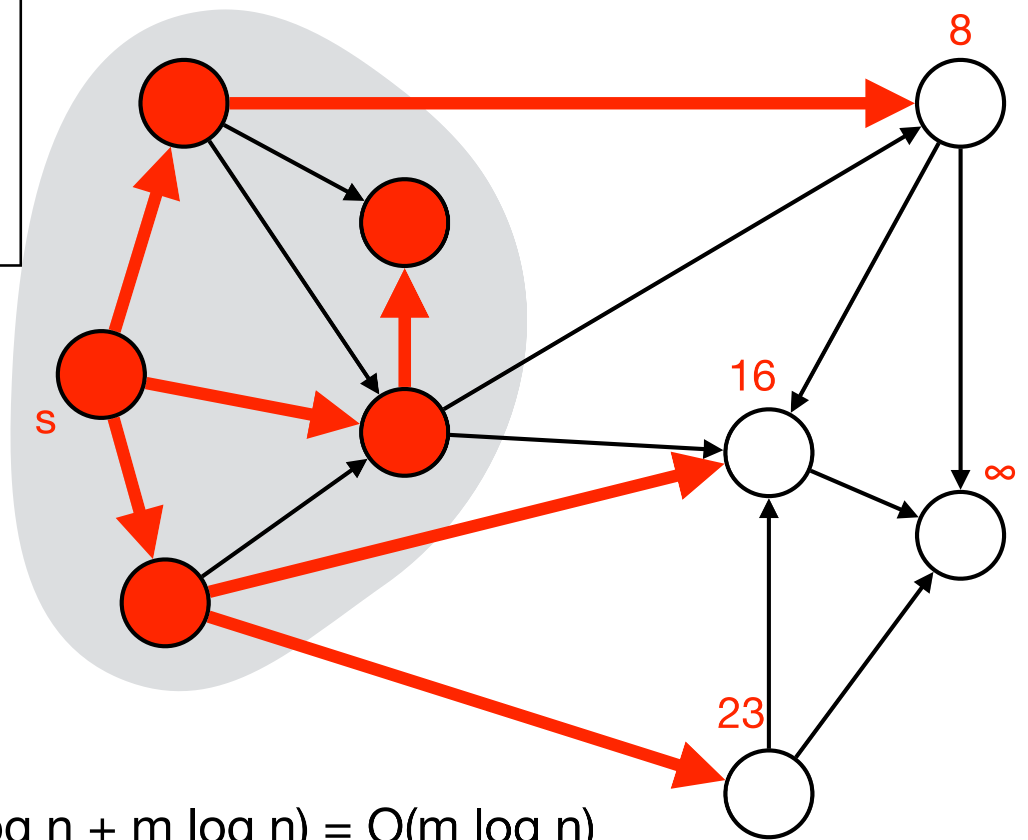
- Implementation. Maintain vertices outside T in priority queue.
  - Key of vertex v = v.d.
  - In each step:
    - Find vertex u with smallest distance estimate = EXTRACT-MIN
    - Relax edges that u point to with DECREASE-KEY.



Dijkstra's Algorithm

```
DIJKSTRA(G, s)
    for all vertices v∈V
        v.d = ∞
        v.π = null
        INSERT(P,v)
    DECREASE-KEY(P,s,0)
    while (P ≠ ∅)
        u = EXTRACT-MIN(P)
        for all v that u point to
            RELAX(u,v)
```

```
RELAX(u,v)
    if (v.d > u.d + w(u,v))
        v.d = u.d + w(u,v)
        DECREASE-KEY(P,v,v.d)
        v.π = u
```



- Time.
    - n EXTRACT-MIN
    - n INSERT
    - < m DECREASE-KEY
- Total time with min-heap. $O(n \log n + n \log n + m \log n) = O(m \log n)$

Dijkstra's Algorithm

- Priority queues and Dijkstra's algorithm. Complexity of Dijkstra's algorithm depend on priority queue.
  - n INSERT
  - n EXTRACT-MIN
  - < m DECREASE-KEY

Priority queue	INSERT	EXTRACT-MIN	DECREASE-KEY	Total
array	$O(1)$	$O(n)$	$O(1)$	$O(n^2)$
binary heap	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci heap	$O(1)^\dagger$	$O(\log n)^\dagger$	$O(1)^\dagger$	$O(m + n \log n)$

† = amortized

- Greed. Dijkstra's algorithm is a greedy algorithm.

Edsger W. Dijkstra

- Edsger Wybe Dijkstra (1930-2002)
- Dijkstra algorithm. "A note on two problems in connexion with graphs". Numerische Mathematik 1, 1959.
- Contributions. Foundations for programming, distributed computation, program verifications, etc.
- Quotes. “*Object-oriented programming is an exceptionally bad idea which could only have originated in California.*”
- “*The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence.*”
- “*APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.*”

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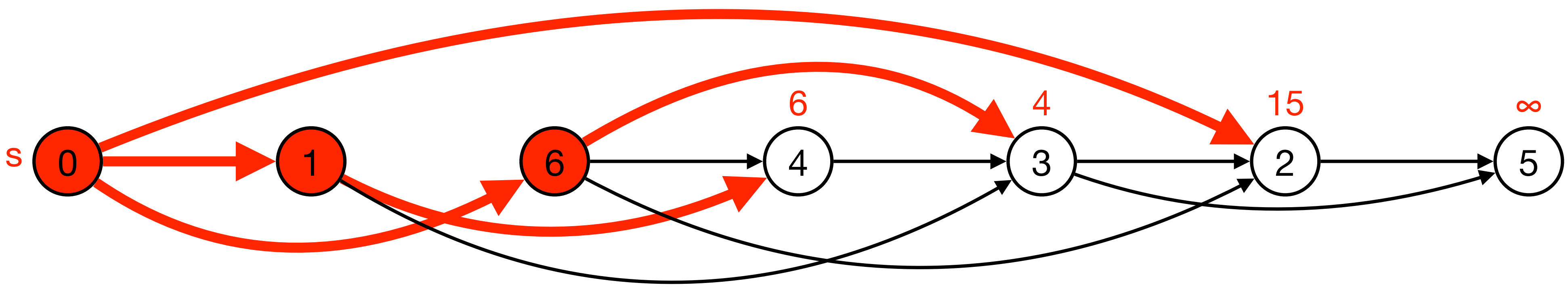
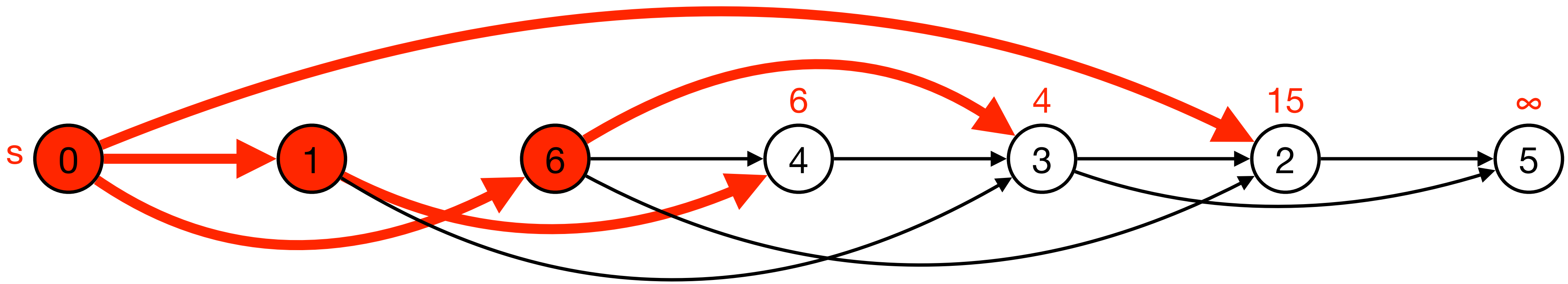
Shortest Paths on DAGs

- Challenge. Is it computationally easier to find shortest paths on DAGs?
- DAG shortest path algoritme.
  - Process vertices in topological order.
  - For each vertex v, relax all edges from v.
- Also works for negative edge weights.



Shortest Paths on DAGs

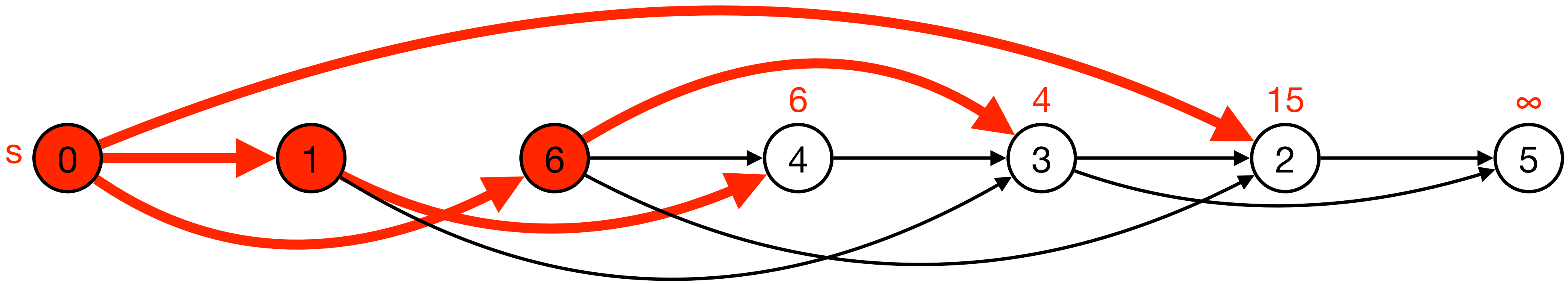
- Lemma. Algorithm computes shortest paths in DAGs.



- Proof.
  - Consider some step after growing tree T and assume distances in T are correct.
  - Consider next vertex u of s not in T.
  - Any path to u consists vertices in T + edge e to u.
  - Edge e is relaxed $\Longrightarrow$ distance to u is shortest.

Shortest Paths on DAGs

- Implementation.
  - Sort vertices in topological order.
  - Relax outgoing edges from each vertex.
- Total time. O(m + n).



Shortest Paths Variants

- Vertices
  - Single source.
  - Single source, single target.
  - All-pairs.

- Edge weights.
  - Non-negative.
  - Arbitrary.
  - Euclidian distances.

- Cycles.
  - No cycles
  - No negative cycles.

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