Priority Queues

- Priority Queues
- Trees and Heaps
- Representations of Heaps
- Algorithms on Heaps
- Building a Heap
- Heapsort

Philip Bille

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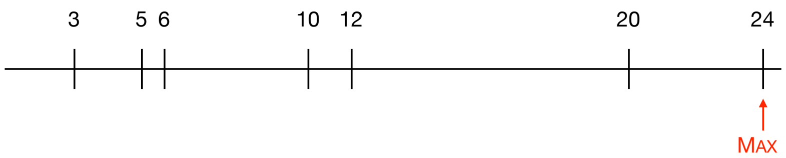
Priority Queues

- Priority queues. Maintain dynamic set S supporting the following operations. Each element has key x.key and satellite data x.data.
  - Max(): return element with largest key.
  - ExtractMax(): return and remove element with largest key.
  - IncreaseKey(x, k): set x.key = k. (assume k ≥ x.key)
  - Insert(x): set S = S ∪ {x}



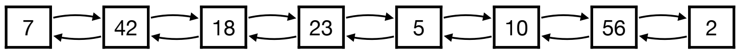
Priority Queues

- Applications.
  - Scheduling
  - Shortest paths in graphs (Dijkstra's algorithm)
  - Minimum spanning trees in graphs (Prim's algorithm)
  - Compression (Huffman's algorithm)
  - …

- Challenge. How can we solve problem with current techniques?

Priority Queues

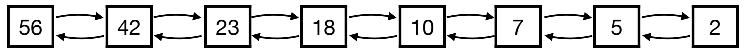
- Solution 1: Linked list. Maintain S in a doubly-linked list.



- Max(): linear search for largest key.
- ExtractMax(): linear search for largest key. Remove and return element.
- IncreaseKey(x, k): set x.key = k.
- Insert(x): add element to front of list (assume element does not exist in S beforehand).
- Time.
  - Max and ExtractMax in O(n) time (n = |S|).
  - IncreaseKey and Insert in O(1) time.
- Space.
  - O(n).

Priority Queues

- Solution 2: Sorted linked list. Maintain S in a sorted doubly-linked list.



- Max(): return first element.
- ExtractMax(): return and remove first element.
- IncreaseKey(x, k): set x.key = k. Linear search to move x to correct position.
- Insert(x): linear search to insert x at correct position.
- Time.
  - Max and ExtractMax in O(1) time.
  - IncreaseKey and Insert in O(n) time.
- Space.
  - O(n).

Priority Queues

Data structure	Max	ExtractMax	IncreaseKey	Insert	Space
linked list	O(n)	O(n)	O(1)	O(1)	O(n)
sorted linked list	O(1)	O(1)	O(n)	O(n)	O(n)

- Challenge. Can we do significantly better?

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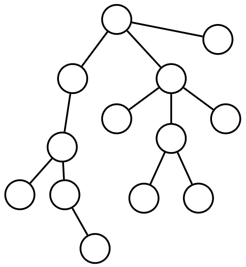
Trees

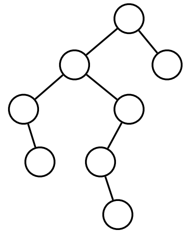
- Rooted trees.
  - Nodes (or vertices) connected with edges.
  - Connected and acyclic.
  - Designated root node.
  - Special type of graph.



- Terminology.
  - Children, parent, descendant, ancestor, leaves, internal nodes, path,..
- Depth and height.
  - Depth of v = length of path from v to root.
  - Height of v = length of path from v to descendant leaf.
  - Depth of T = height of T = length of longest path from root to a leaf.

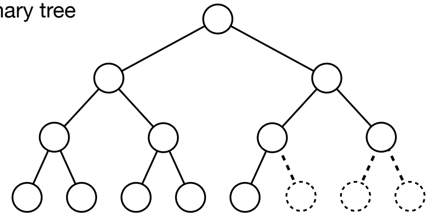
Trees

- Binary tree.
  - Rooted tree.
  - Each node has at most two children called the left child and right child



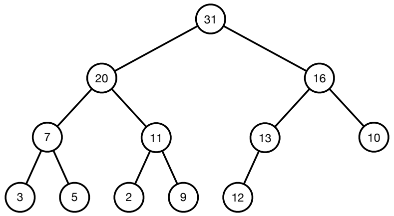
- Complete binary tree. Binary tree where all levels of tree are full.
- Almost complete binary tree. Complete binary tree with 0 or more rightmost leaves deleted.

- Lemma. Height af an (almost) complete binary tree with n nodes is Θ(log n).
- Pf. See exercises.



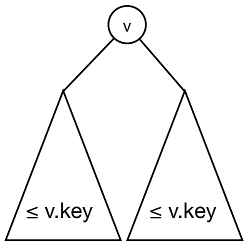
Heaps

- Heaps. Almost complete binary tree. All nodes store one element and the tree satisfies heap-order.



- Heap-order.
  - For all nodes v:
    - all keys in left subtree and right subtree are ≤ v.key.

- Max-heap vs min-heap.



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Heap

- Data structure. We need the following navigation operations on a heap.
  - PARENT(x): return parent of x.
  - LEFT(x) : return left child of x.
  - RIGHT(x): return right child of x.

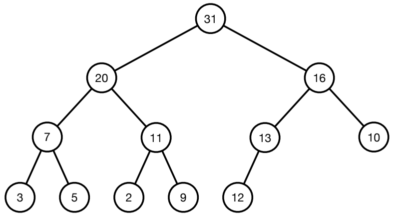
- Challenge. How can we represent a heap compactly to support fast navigation?

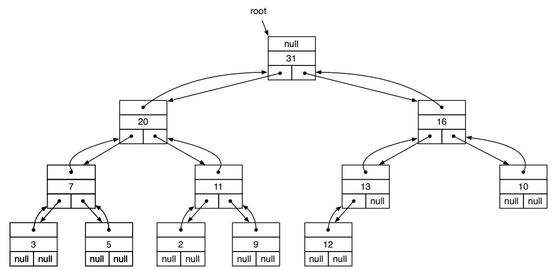
Heap

- Linked representation. Each node stores
  - v.key
  - v.parent
  - v.left
  - v.right

- PARENT, LEFT, RIGHT by following pointer.

- Time. O(1)
- Space. O(n)



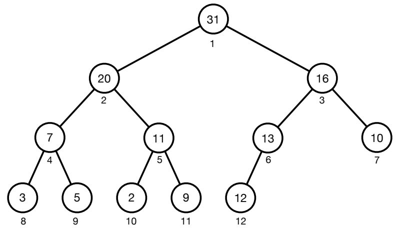


Heap

- Array representation.
  - Array H[0..n]
  - H[0] unused
  - H[1..n] stores nodes in level order.

- PARENT(x): return $\lfloor x/2 \rfloor$
- LEFT(x) : return 2x.
- RIGHT(x): return 2x + 1

- Time. O(1)
- Space. O(n)



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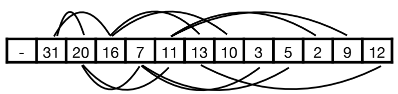
-	31	20	16	7	11	13	10	3	5	2	9	12



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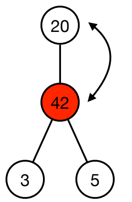
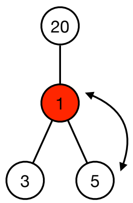
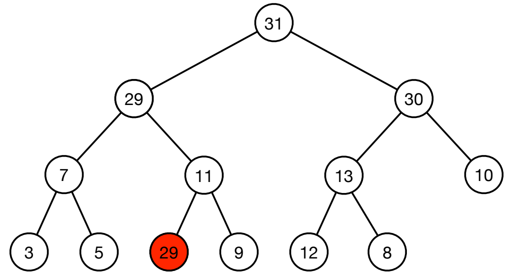
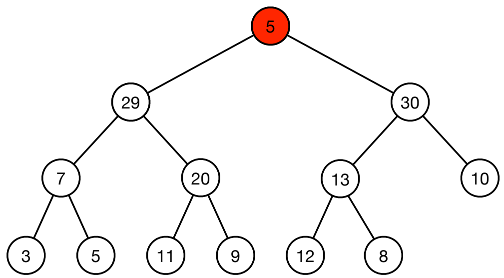
Algorithms on Heaps

- BubbleUp(x):
  - If heap order is violated at node x because key is larger than key at parent:
  - Swap x and parent
  - Repeat with parent until heap order is satisfied.
- BubbleDown(x):
  - If heap order is violated at node x because key is smaller than key at left or right child:
  - Swap x and child c with largest key.
  - Repeat with child until heap order is satisfied.



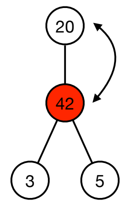


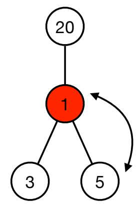




Algorithms on Heaps

- BubbleUp(x):
  - If heap order is violated at node x because key is larger than key at parent:
  - Swap x and parent
  - Repeat with parent until heap order is satisfied.
- BubbleDown(x):
  - If heap order is violated at node x because key is smaller than key at left or right child:
  - Swap x and child c with largest key.
  - Repeat with child until heap order is satisfied.
- Time.
  - BubbleUp and BubbleDown in O(log n) time.
- How can we use them to implement a priority queue?





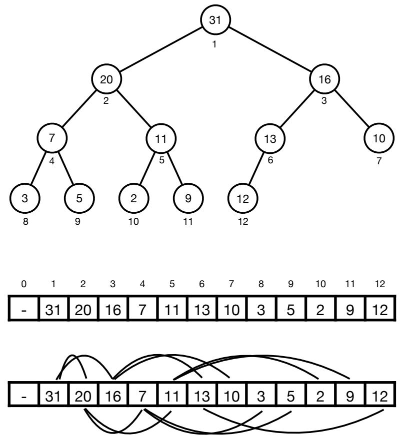
Priority Queues

```
MAX()
   return H[1]
```

```
INSERT(x)
   n = n + 1
   H[n] = x
   BUBBLEUP(n)
```

```
EXTRACTMAX()
   r = H[1]
   H[1] = H[n]
   n = n - 1
   BUBBLEDOWN(1)
   return r
```

```
INCREASEKEY(x,k)
   H[x] = k
   BUBBLEUP(x)
```



- Ex. Trace execution of following sequence in initially empty heap: 2, 5, 7, 6, 4, E, E
- Numbers mean INSERT og E is EXTRACTMAX. Draw heap after each operation.

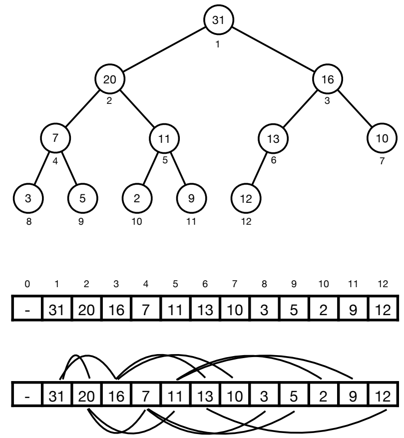
Priority Queues

```
MAX()
   return H[1]
```

```
INSERT(x)
   n = n + 1
   H[n] = x
   BUBBLEUP(n)
```

```
EXTRACTMAX()
   r = H[1]
   H[1] = H[n]
   n = n - 1
   BUBBLEDOWN(1)
   return r
```

```
INCREASEKEY(x,k)
   H[x] = k
   BUBBLEUP(x)
```



- Time.
  - MAX in O(1) time.
  - EXTRACTMAX, INCREASEKEY, and INSERT in O(log n) time.

Priority Queues

Data structure	MAX	EXTRACTMAX	INCREASEKEY	INSERT	Space
linked list	O(n)	O(n)	O(1)	O(1)	O(n)
sorted linked list	O(1)	O(1)	O(n)	O(n)	O(n)
heap	O(1)	O(log n)	O(log n)	O(log n)	O(n)

- Heaps with array data structure is an example of an implicit data structure.

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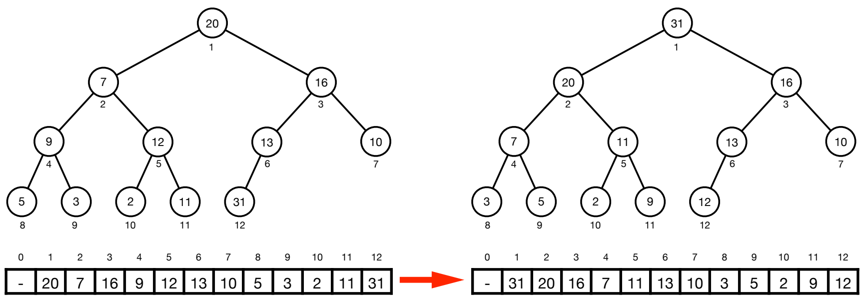
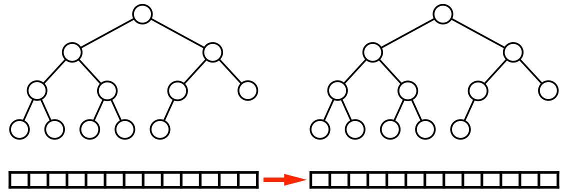
Building a Heap

- Building a heap. Given n integers in a array H[1..n], convert array to a heap.



Building a Heap

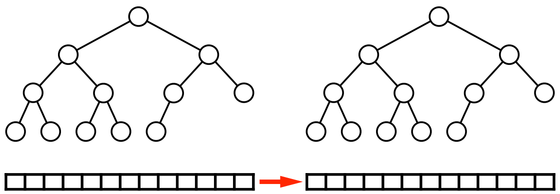
- Solution 1: top-down construction
  - For all nodes in increasing level order apply BUBBLEUP.



- Time.
  - For each node of depth d, we use O(d) time.
  - 1 node of depth 0, 2 nodes of depth 1, 4 nodes of depth 2, ..., ~n/2 nodes of depth log n.
  - ⇒ total time is O(n log n)
- Challenge. Can we do better?

Building a Heap

- Solution 2: bottom-up construction
  - For all nodes in decreasing level order apply BUBBLEDOWN.



- Time.
  - For each node of height h we use O(h) time.
  - 1 node of height log n, 2 nodes of height log n - 1, ..., n/4 nodes of height 1, n/2 nodes of height 0.
  - ⇒ total time is O(n) (see exercise)

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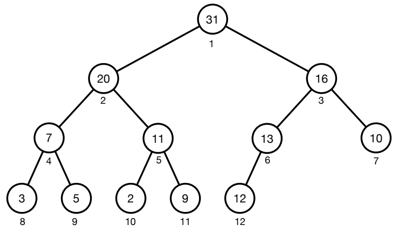
Heapsort

- Sorting. How can we sort an array H[1..n] using a heap?
- Solution.
  - Build a heap for H.
  - Apply n EXTRACTMAX.
    - Insert results in the end of array.
  - Return H.



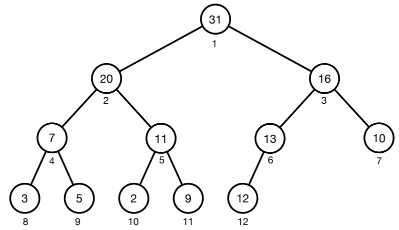
0	1	2	3	4	5	6	7	8	9	10	11	12
-	31	20	16	7	11	13	10	3	5	2	9	12

- Time.
  - Heap construction in O(n) time.
  - n EXTRACTMAX in O(nlog n) time.
  - ⇒ total time is O(nlog n).

Heapsort

- Theorem. We can sort an array in O(n log n) time.

- Uses only O(1) extra space.
- In-place sorting algorithm.



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