Search Trees

- Dynamic Ordered Sets
- Binary Search Trees
- Balanced Search Trees

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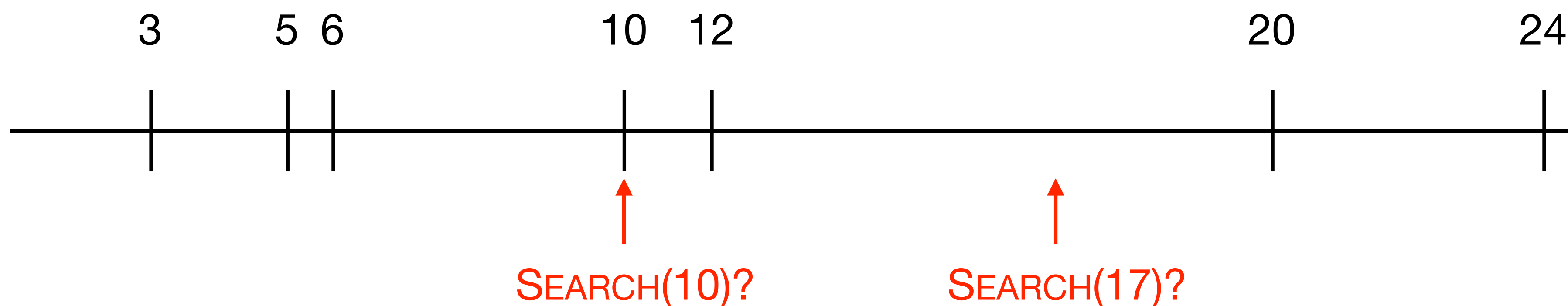
Dynamic Ordered Sets

- Dynamic Ordered Sets. Maintain dynamic set S supporting the following operations. Each element x has key x.key and satellite data x.data.
  - SEARCH(k): return element x such that x.key = k if it exists. Otherwise return null.
  - INSERT(x): add x to S (assume x.key is not already in S).
  - DELETE(x): remove x from S.
- We want to maintain elements ordered by the keys. Allows efficient support for many other important operations and other features.



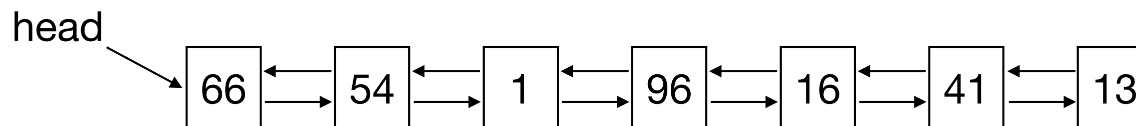
Dynamic Ordered Sets

- Applications.
  - Dictionaries.
  - Indexes.
  - Filesystem.
  - Databases.
  - ....

- Challenge. How can we solve problem with current techniques?

Dynamic Ordered Sets

- Solution 1: linked list. Maintain S in a doubly-linked list.



- SEARCH(k): linear search for largest key ≤ k.
- INSERT(x): insert x in the front of list.
- DELETE(x): remove x from list.

- Time.
  - SEARCH in O(n) time (n = |S|).
  - INSERT and DELETE in O(1) time.
- Space.
  - O(n).

Dynamic Ordered Sets

- Solution 2: sorted array. Maintain S in an sorted array according to keys.

1	2	3	4	5	6	7
1	13	16	41	54	66	96

- SEARCH(k): binary search for k.
- INSERT(x): find index using SEARCH(x.key). Build new array of size +1 with x inserted.
- DELETE(x): build new array of size -1 with element with key k removed.

- Time.
  - SEARCH in O(log n) time.
  - INSERT and DELETE in O(n) time.
- Space.
  - O(n).

Nearest Neighbor

Data structure	SEARCH	INSERT	DELETE	Space
linked list	O(n)	O(1)	O(1)	O(n)
sorted array	O(log n)	O(n)	O(n)	O(n)

- Challenge. Can we do significantly better?

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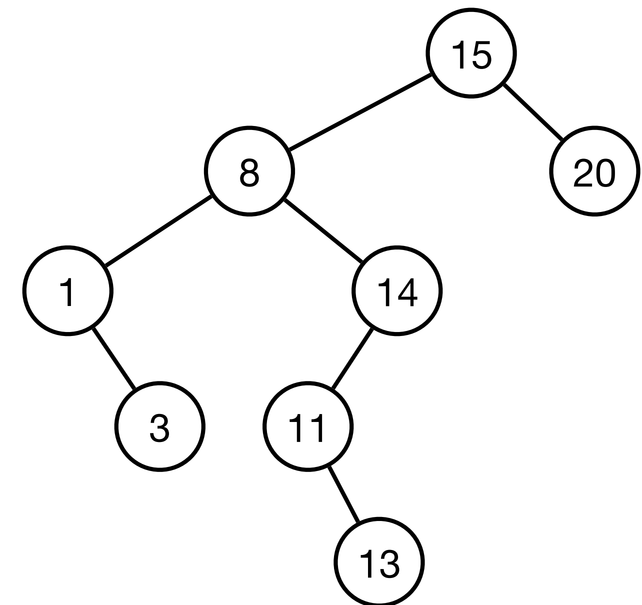
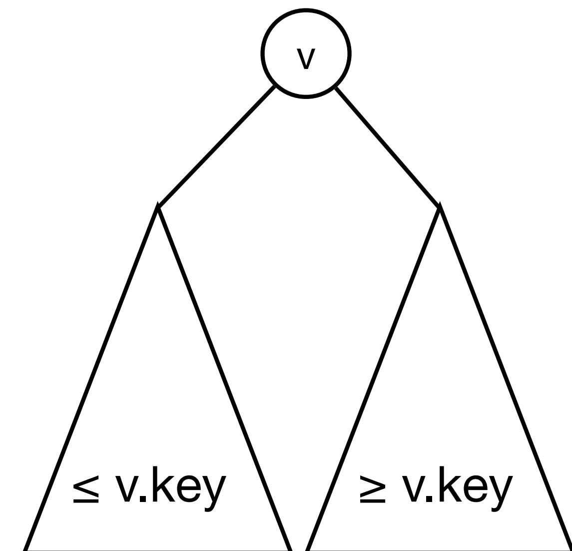
Binary Search Trees

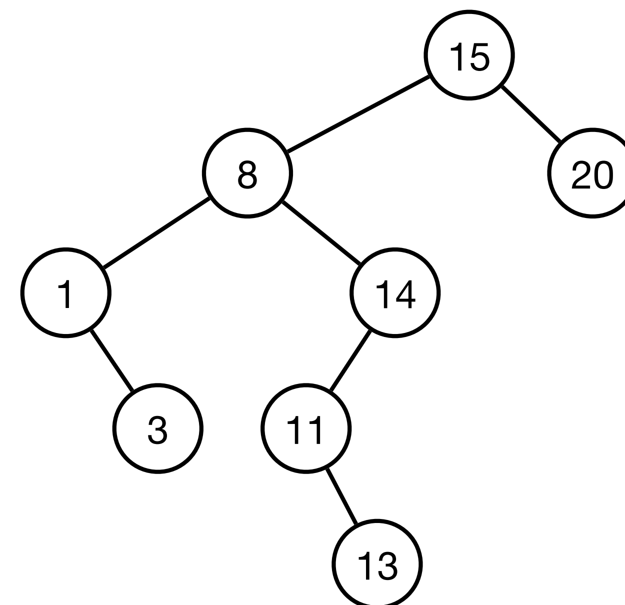
- Binary tree.
  - Rooted tree
  - Each internal node has a left child and/or a right child.
- Binary search tree.
  - Binary tree in symmetric order.
- Symmetric order. For each vertex v:
  - all vertices in left subtree are < v.key.
  - all vertices in right subtree are > v.key.





Binary Search Trees

- Symmetric order ~ inorder traversal outputs the keys in sorted order.



- Inorder traversal.
  - Visit left subtree recursively.
  - Visit vertex.
  - Visit right subtree recursively.
- Preorder traversal.
  - Visit vertex.
  - Visit left subtree recursively.
  - Visit right subtree recursively.
- Postorder traversal.
  - Visit left subtree recursively.
  - Visit right subtree recursively.
  - Visit vertex.

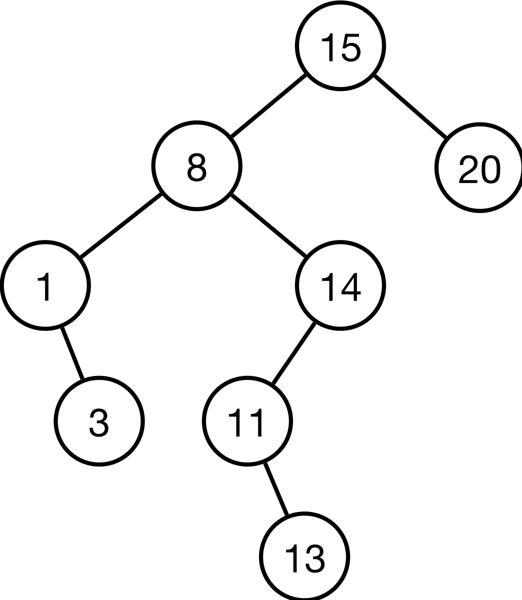
Inorder: 1, 3, 8, 11, 13, 14, 15, 20

Preorder: 15, 8, 1, 3, 14, 11, 13, 20

Postorder: 3, 1, 13, 11, 14, 8, 20, 15

Binary Search Trees

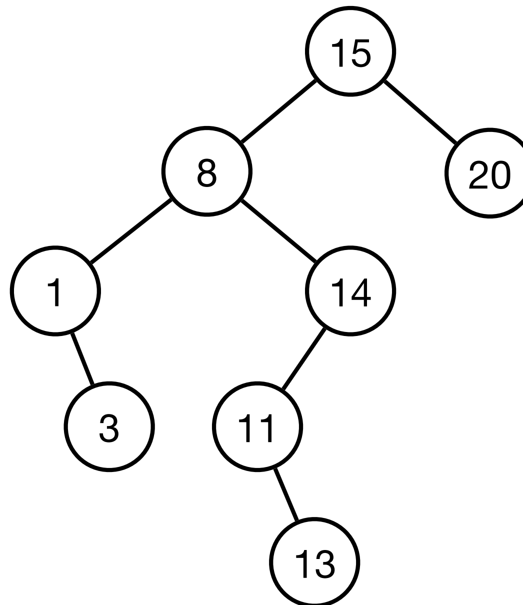
- Representation. Each node x stores
  - x.key
  - x.left
  - x.right
  - x.parent
  - (x.data)
- Space. O(n)



Binary Search Trees

- SEARCH(k): traverse tree top-down.
  - Compare key k against key in node.
  - If equal return element. If less go left. If greater go right.
  - If we reach bottom, return null.

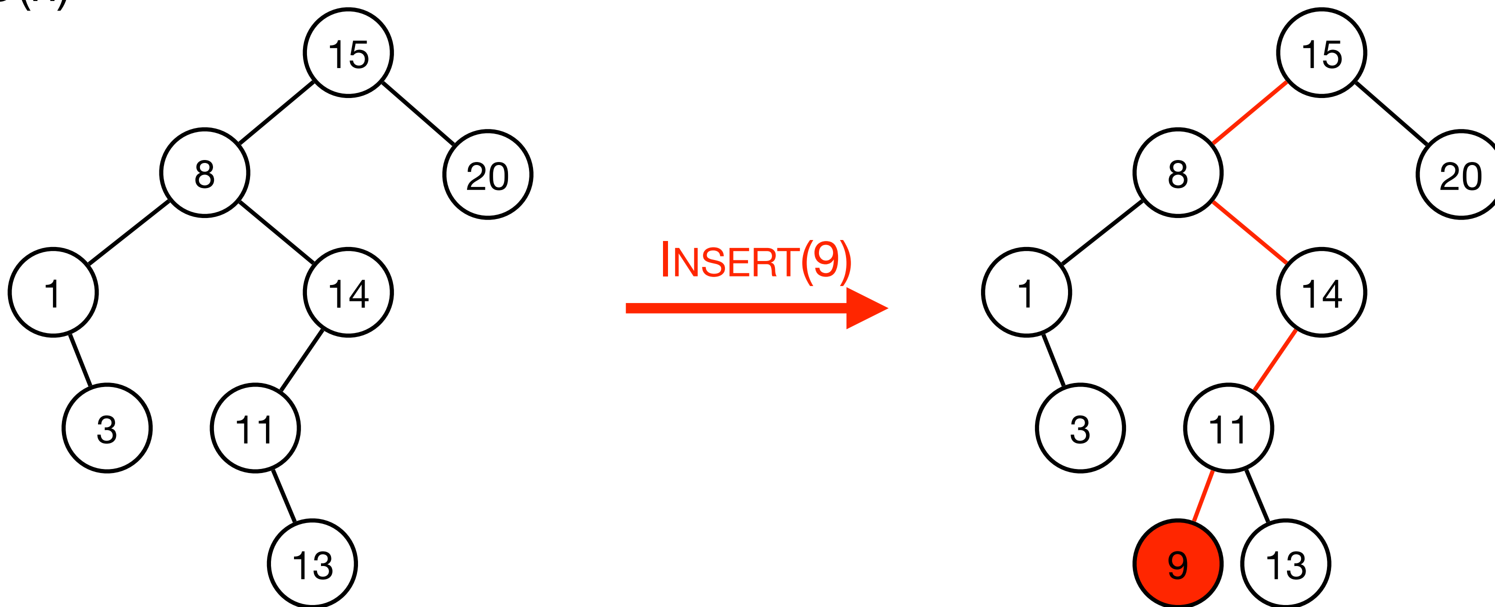
- Time. O(h)



Binary Search Trees

- INSERT(x): traverse tree top-down and compare keys.
  - search for x.
  - add x at leaf.

- Time. O(h)



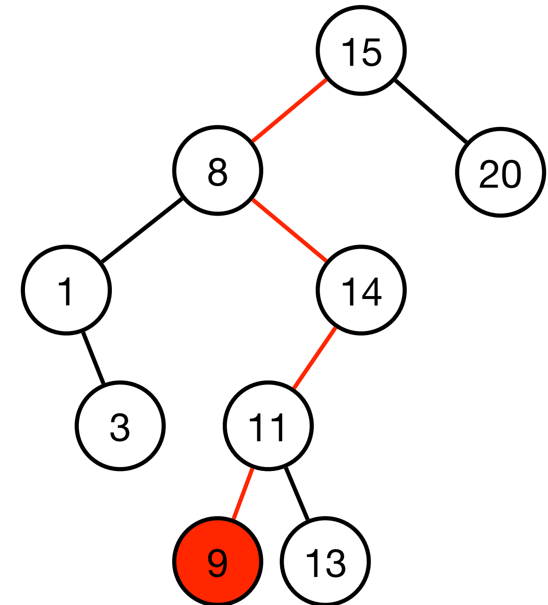
Binary Search Trees

- INSERT(x): traverse tree top-down and compare keys.
  - if less go left; if greater go right; if equal, return node.
  - if null, insert x.

- Exercise. Insert following sequence in binary search tree: 6, 14, 3, 8, 12, 9, 34, 1, 7

Binary Search Trees

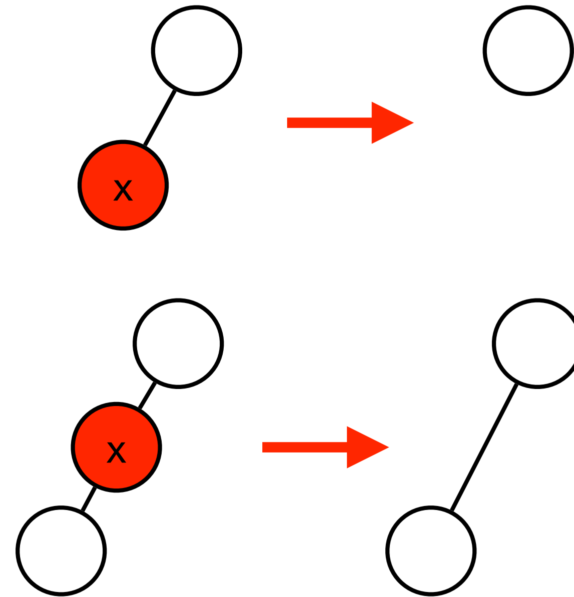
```
INSERT(x,v)
   if (v == null) return x
   if (x.key ≤ v.key)
      v.left = INSERT(x, v.left)
   if (x.key > v.key)
      v.right = INSERT(x, v.right)
```

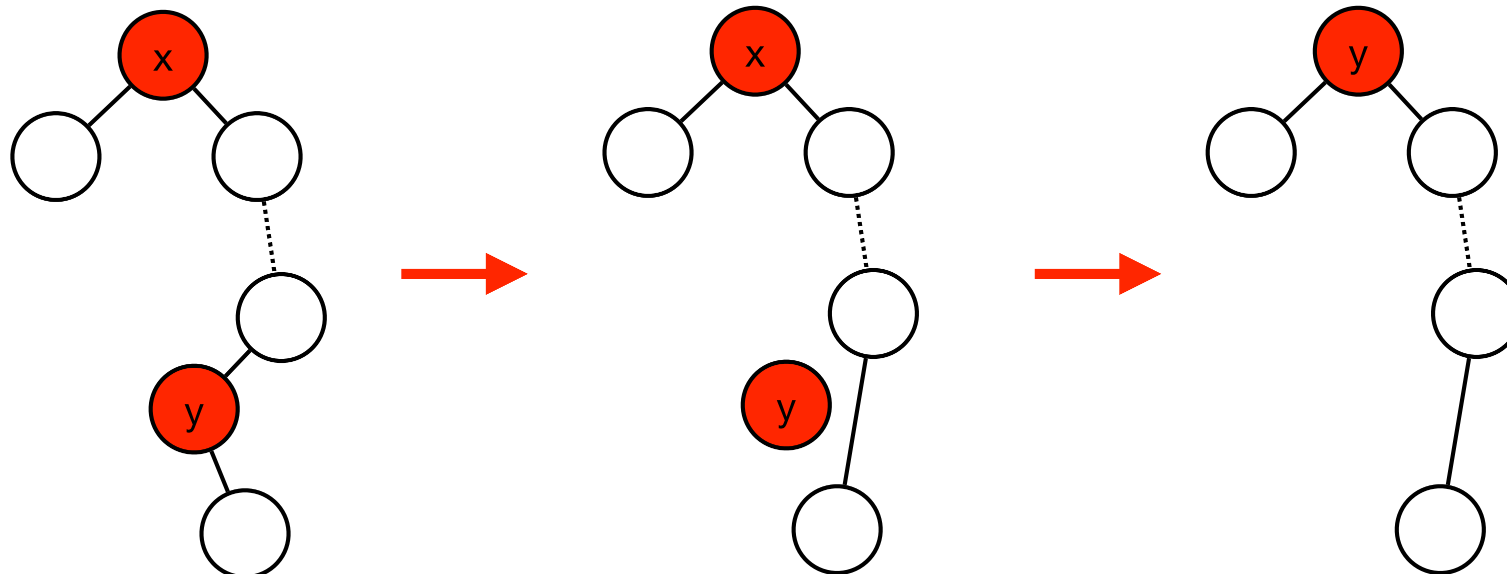


- Time. O(h)

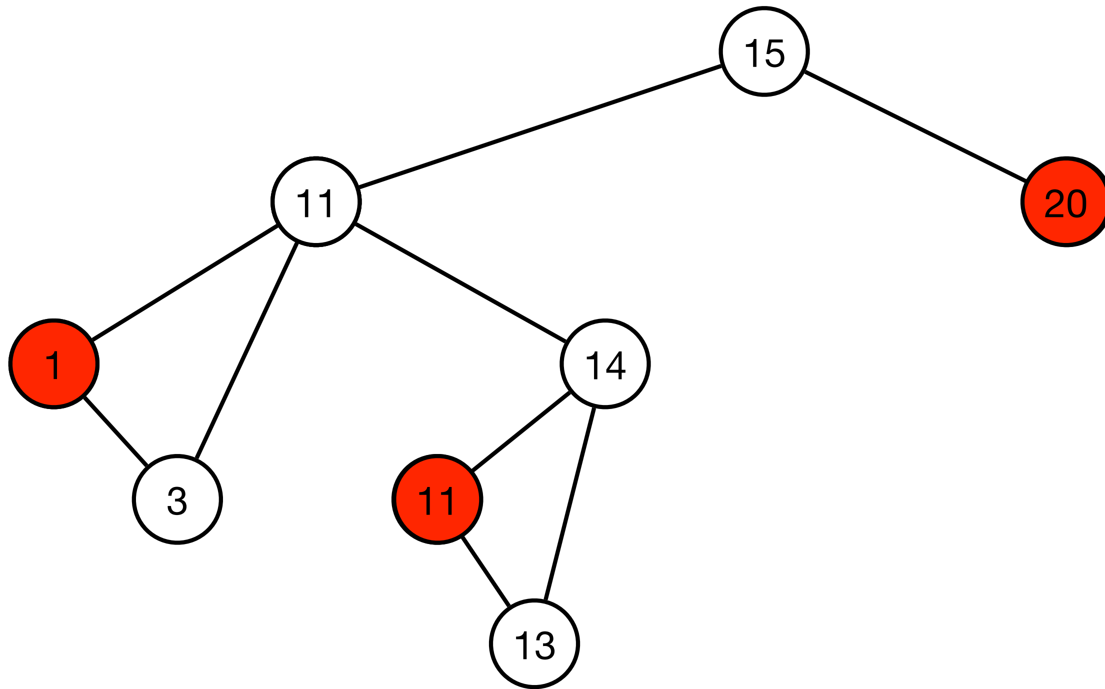
Binary Search Trees

- DELETE(x):
  - 0 children: remove x.
  - 1 child: splice x.
  - 2 children: find y = node with smallest key > x.key. Splice y and replace x by y.





DELETE 20 1 8



Binary Search Trees

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- Time. O(h)

Dynamic Ordered Sets

Data structure	SEARCH	INSERT	DELETE	Space
linked list	O(n)	O(1)	O(1)	O(n)
sorted array	O(log n)	O(n)	O(n)	O(n)
binary search tree	O(h)	O(h)	O(h)	O(n)

- Height. Depends on sequence of operations.
  - $h = \Omega(n)$ worst-case and $h = \Theta(\log n)$ on average.
- Challenge. Can we maintain height at $O(\log n)$ worst-case?

Search Trees

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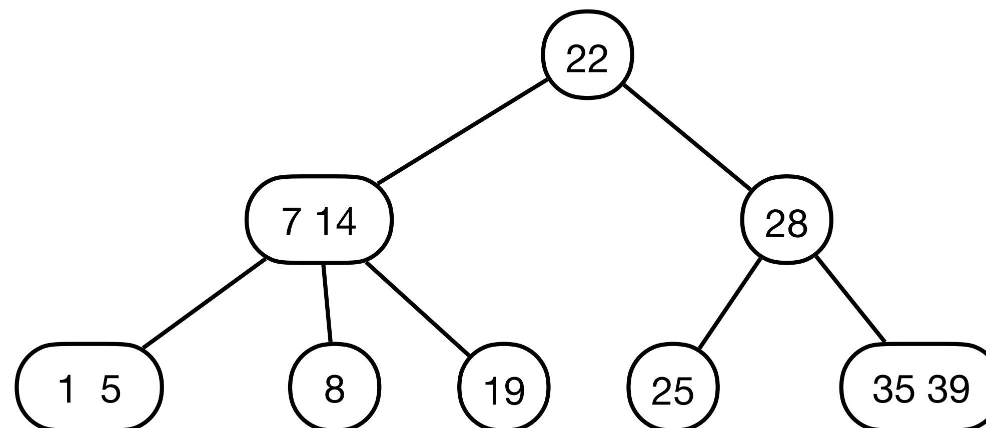
- Balanced Search Trees

Balanced Search Trees

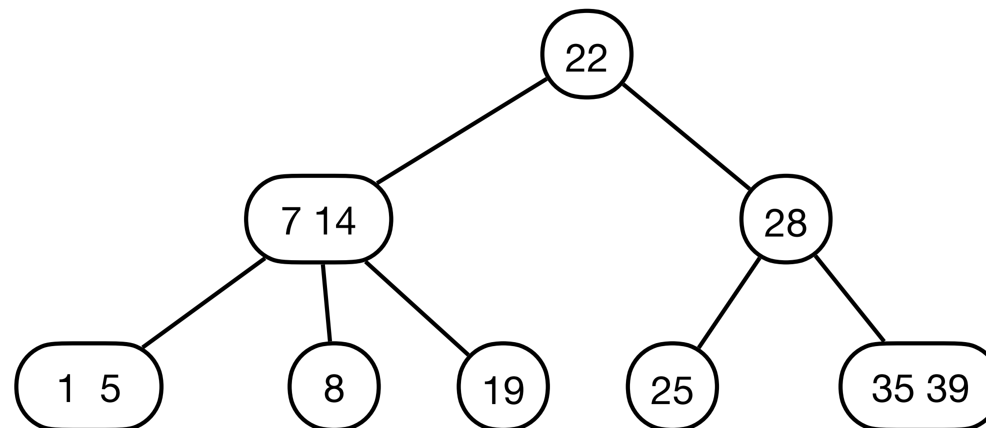
- 2-3 Tree.
  - Rooted tree.
  - Each internal node has 2 or 3 children.
    - 2-node: 2 children and 1 key
    - 3-node: 3 children and 2 keys.
  - Symmetric order.
    - Inorder traversal outputs the keys in sorted order.
  - Perfect balance.
    - Every path from root to a leaf has the same length
  - ⇒ height of tree is $\Theta(\log n)$



Balanced Search Trees

- SEARCH(k): traverse tree top-down.
  - Compare key k against keys in node.
  - If equal return element. Otherwise, recurse in child with interval containing k and recurse.
  - If we reach bottom, return null.

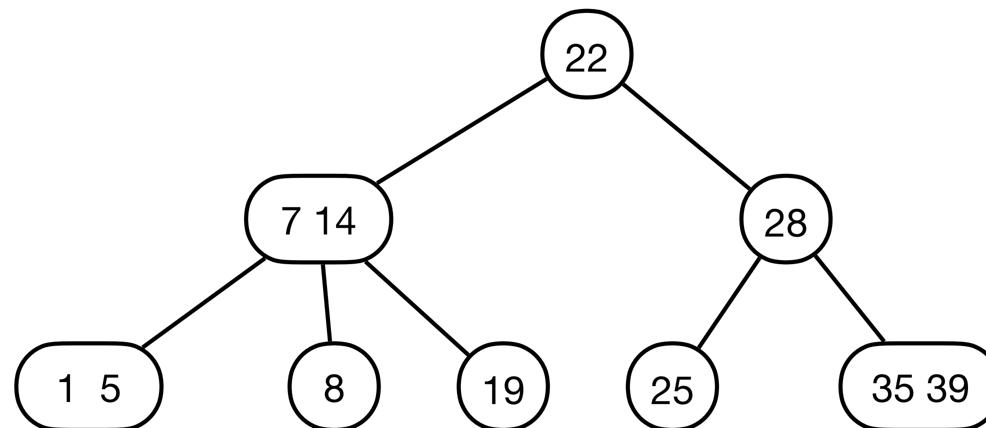
- Time. O(log n)



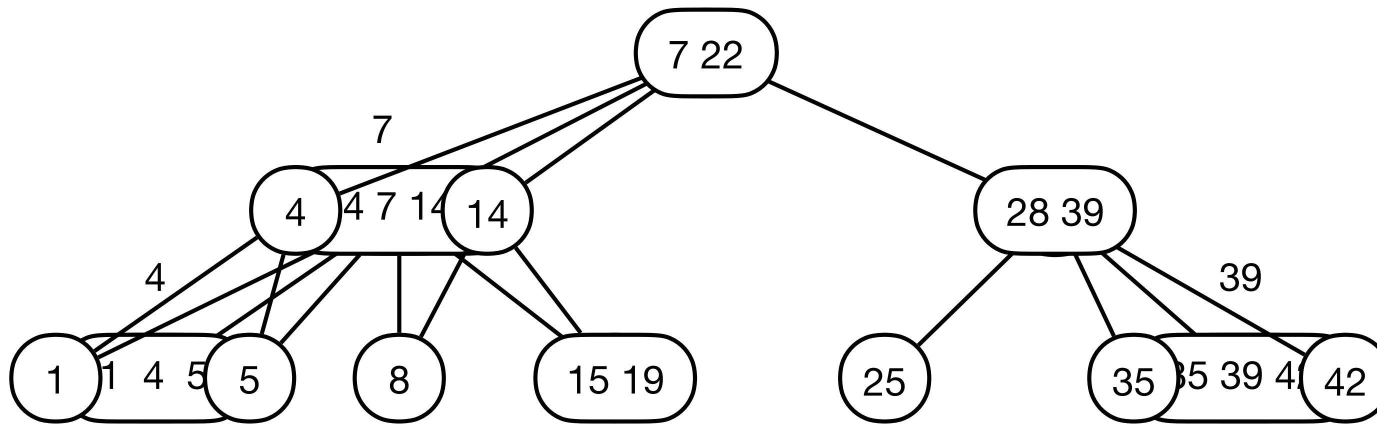
Balanced Search Trees

- INSERT(x):
  - Search for x.
  - Add x at leaf.
  - If too large, move middle key to parent. Repeat if necessary.

- Time. O(log n)



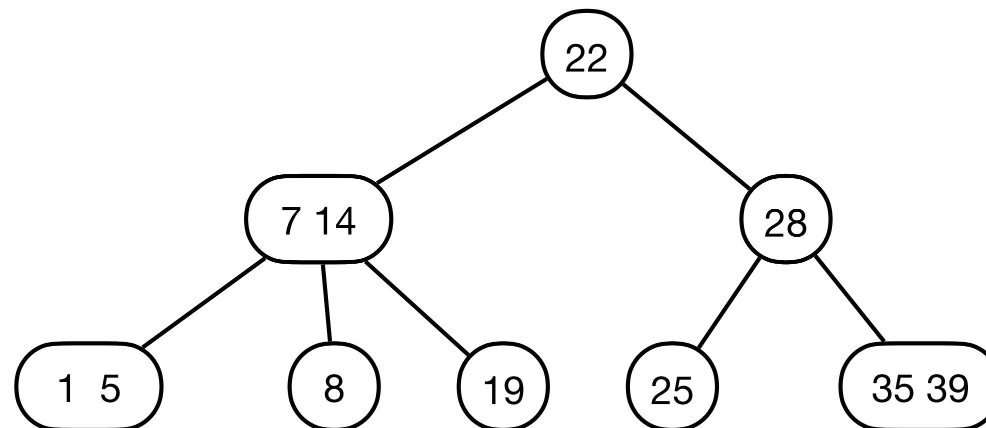
INSERT 15 42 4



Balanced Search Trees

- DELETE(x):
  - Search for x.
  - If x is a not a leaf, find node with smallest key > x.key, swap with x, and delete it.
  - If too small, take from parent. Repeat if necessary.

- Time. O(log n)



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linked list	O(n)	O(1)	O(1)	O(n)
sorted array	O(log n)	O(n)	O(n)	O(n)
binary search tree	O(h)	O(h)	O(h)	O(n)
2-3 tree	O(log n)	O(log n)	O(log n)	O(n)

- Can we do better?
  - Many variants of balanced search trees supporting many different operations.
  - Many efficient practical solutions.
  - Optimal time bounds for comparison-based data structures.
  - Even better bounds possible with more advanced techniques.

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