Search Trees

- Dynamic Ordered Sets
- Binary Search Trees
- Balanced Search Trees

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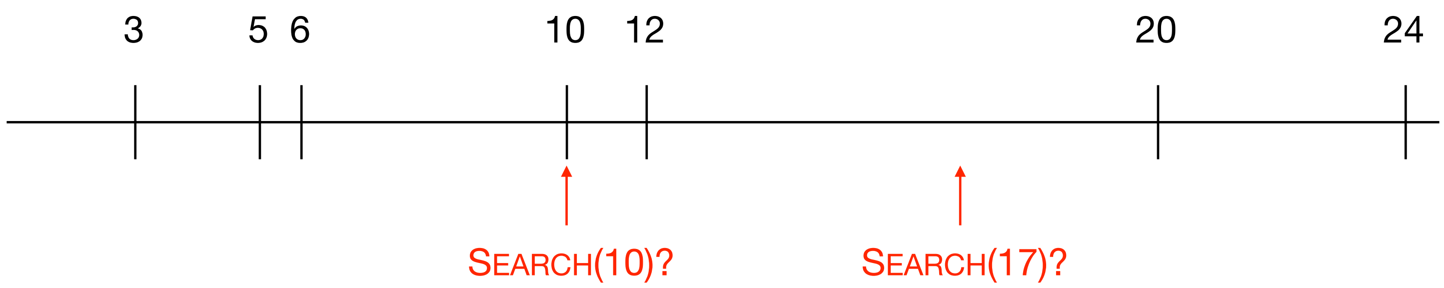
- Balanced Search Trees

Dynamic Ordered Sets

- Dynamic Ordered Sets. Maintain dynamic set S supporting the following operations. Each element x has key x.key and satellite data x.data.
  - SEARCH(k): return element x such that x.key = k if it exists. Otherwise return null.
  - INSERT(x): add x to S (assume x.key is not already in S).
  - DELETE(x): remove x from S.
- We want to maintain elements ordered by the keys. Allows efficient support for many other important operations and other features.



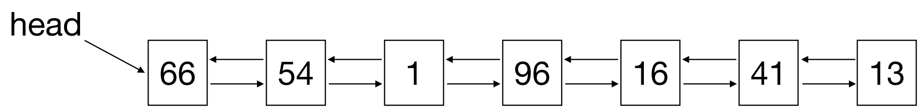
Dynamic Ordered Sets

- Applications.
  - Dictionaries.
  - Indexes.
  - Filesystem.
  - Databases.
  - ....

- Challenge. How can we solve problem with current techniques?

Dynamic Ordered Sets

- Solution 1: linked list. Maintain S in a doubly-linked list.



- SEARCH(k): linear search for k.
- INSERT(x): insert x in the front of list.
- DELETE(x): remove x from list.

- Time.
  - SEARCH in O(n) time (n = |S|).
  - INSERT and DELETE in O(1) time.
- Space.
  - O(n).

Dynamic Ordered Sets

- Solution 2: sorted array. Maintain S in an sorted array according to keys.

1	2	3	4	5	6	7
1	13	16	41	54	66	96

- SEARCH(k): binary search for k.
- INSERT(x): find index using SEARCH(x.key). Build new array of size +1 with x inserted.
- DELETE(x): build new array of size -1 with element with key k removed.

- Time.
  - SEARCH in O(log n) time.
  - INSERT and DELETE in O(n) time.
- Space.
  - O(n).

Dynamic Ordered Sets

Data structure	SEARCH	INSERT	DELETE	Space
linked list	O(n)	O(1)	O(1)	O(n)
sorted array	O(log n)	O(n)	O(n)	O(n)

- Challenge. Can we do significantly better?

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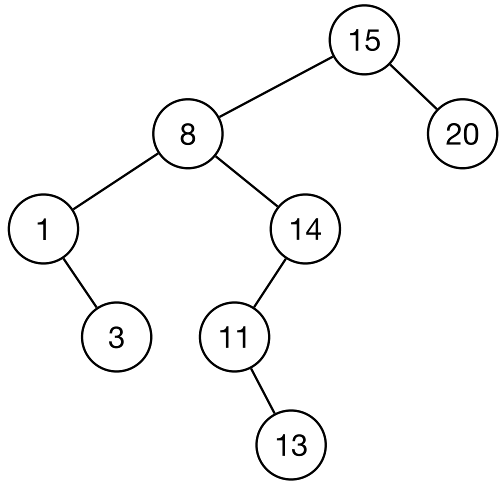
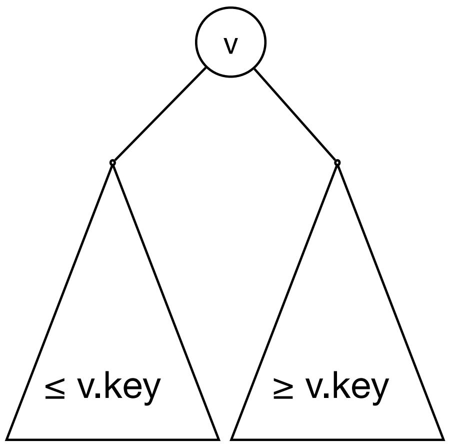
Binary Search Trees

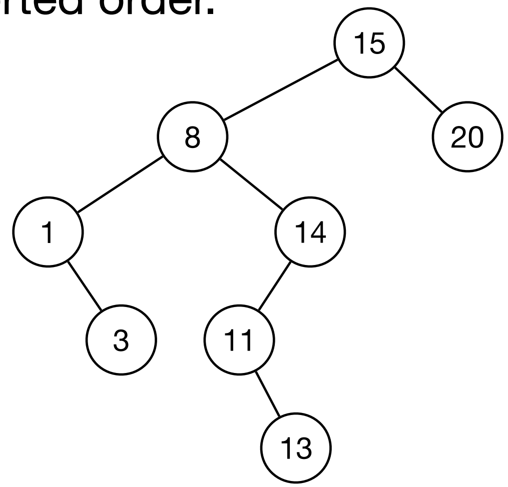
- Binary tree.
  - Rooted tree
  - Each internal node has a left child and/or a right child.
- Binary search tree.
  - All nodes store an element.
  - Elements are in symmetric order.
- Symmetric order. For each vertex v:
  - all vertices in left subtree are ≤ v.key.
  - all vertices in right subtree are ≥ v.key.





Binary Search Trees

- Symmetric order ~ inorder traversal outputs the keys in sorted order.



- Inorder traversal.
  - Visit left subtree recursively.
  - Visit vertex.
  - Visit right subtree recursively.
- Preorder traversal.
  - Visit vertex.
  - Visit left subtree recursively.
  - Visit right subtree recursively.
- Postorder traversal.
  - Visit left subtree recursively.
  - Visit right subtree recursively.
  - Visit vertex.

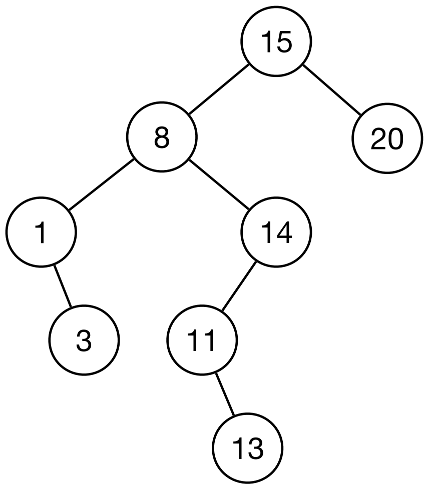
Inorder: 1, 3, 8, 11, 13, 14, 15, 20

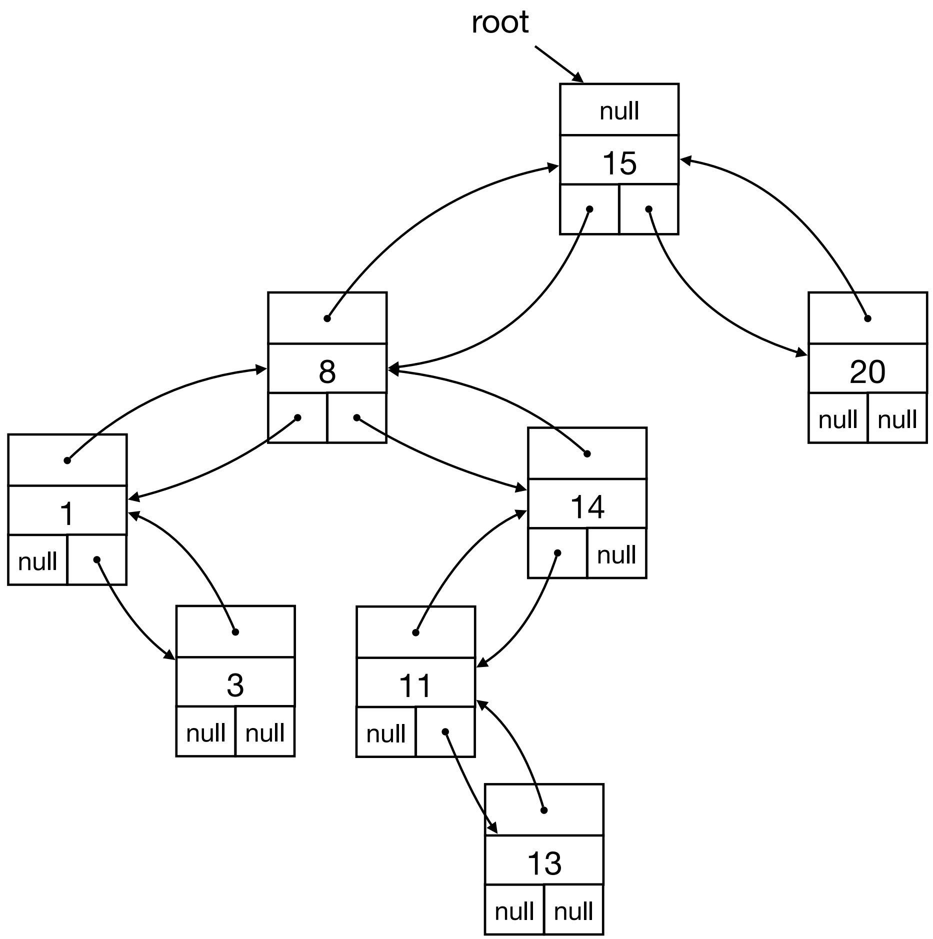
Preorder: 15, 8, 1, 3, 14, 11, 13, 20

Postorder: 3, 1, 13, 11, 14, 8, 20, 15

Binary Search Trees

- Representation. Each node x stores
  - x.key
  - x.left
  - x.right
  - x.parent
  - (x.data)
- Space. O(n)

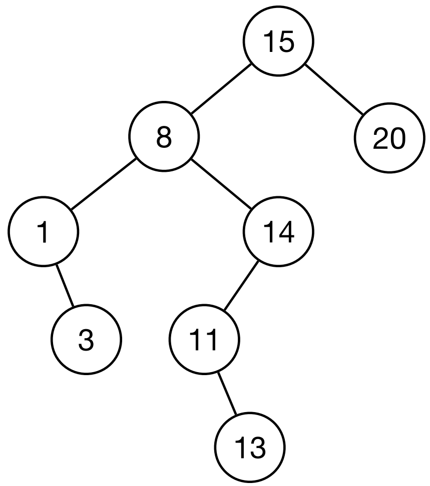




Binary Search Trees

- SEARCH(k): traverse tree top-down.
  - Compare key k against key in node.
  - If equal return element. If less go left. If greater go right.
  - If we reach bottom, return null.

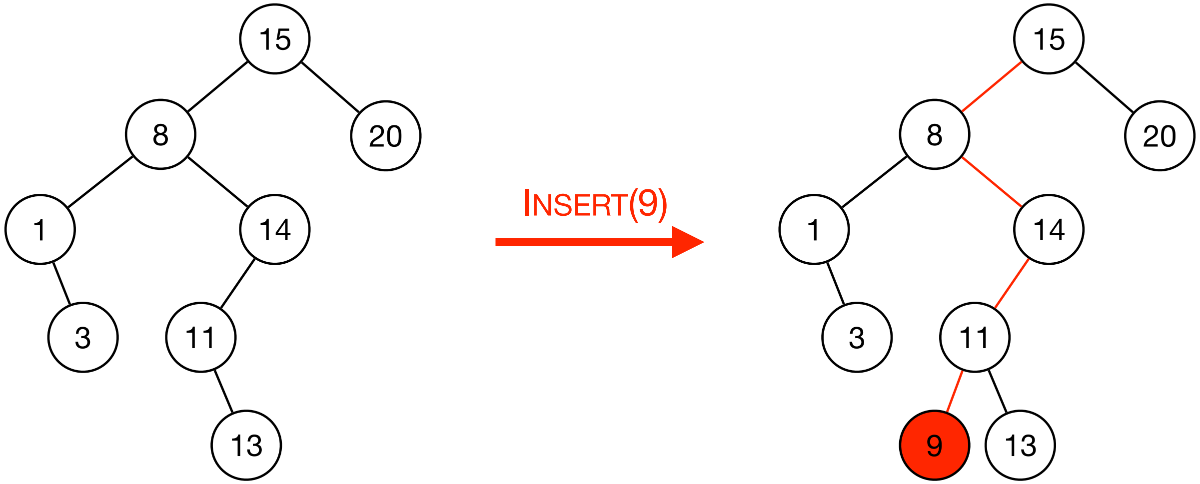
- Time. O(h)



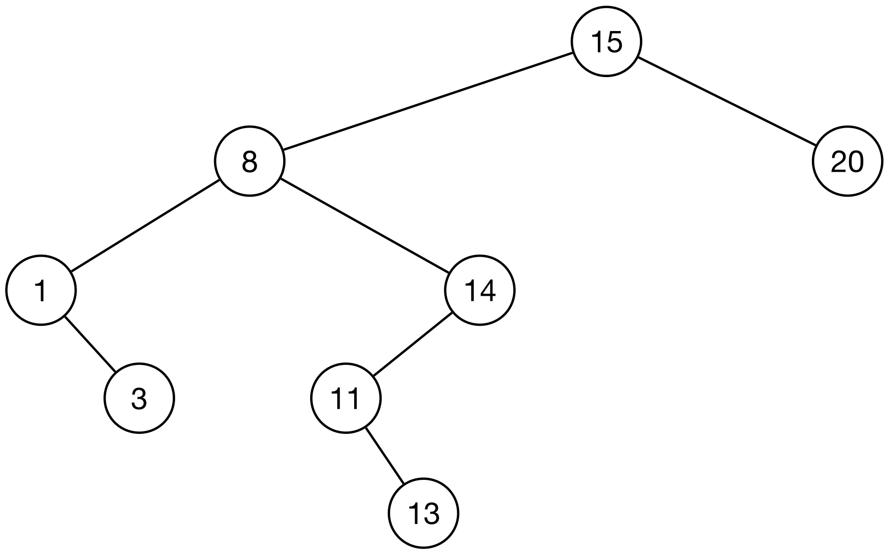
Binary Search Trees

- INSERT(x): traverse tree top-down and compare keys.
  - search for x.
  - add x at leaf.

- Time. O(h)



INSERT 15 8 20 14 1 3 11 13



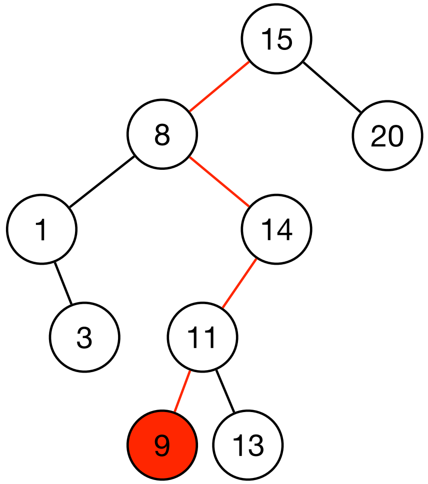
Binary Search Trees

- INSERT(x): traverse tree top-down and compare keys.
  - if less go left; if greater go right; if equal, return node.
  - if null, insert x.

- Exercise. Insert following sequence in binary search tree: 6, 14, 3, 8, 12, 9, 34, 1, 7

Binary Search Trees

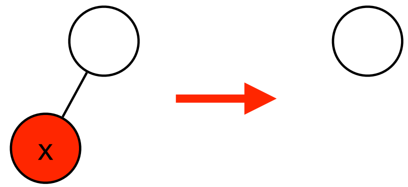
```
INSERT(x,v)
    if (v == null) return x
    if (x.key ≤ v.key)
        v.left = INSERT(x, v.left)
    if (x.key > v.key)
        v.right = INSERT(x, v.right)
```

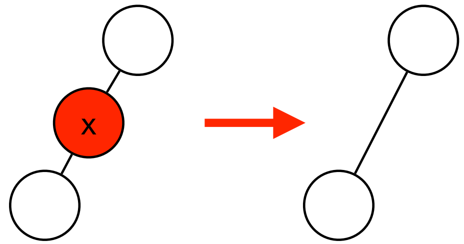


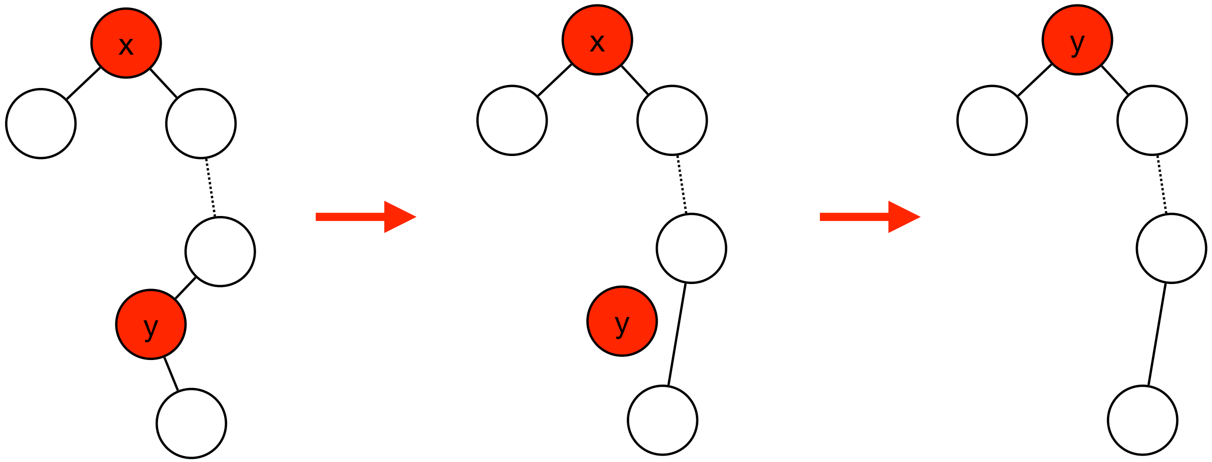
- Time. O(h)

Binary Search Trees

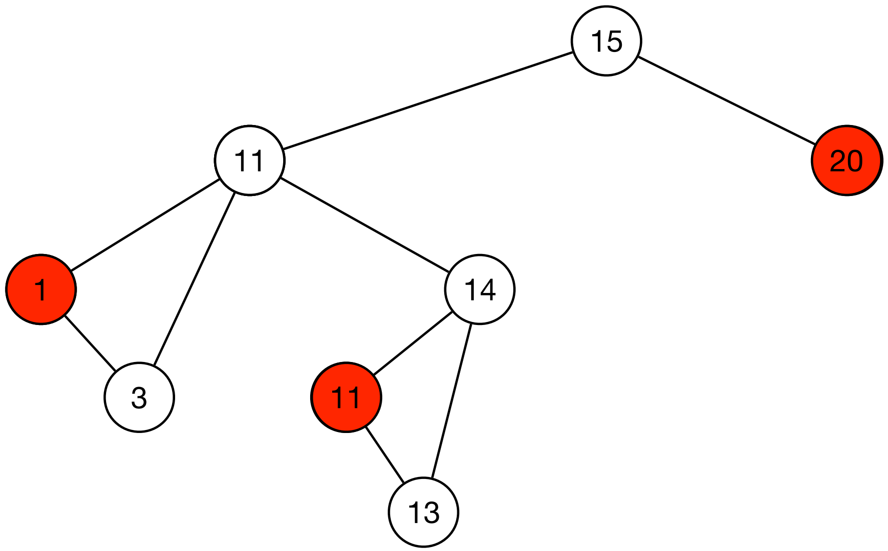
- DELETE(x):
  - 0 children: remove x.
  - 1 child: splice x.
  - 2 children: find y = node with smallest key > x.key. Splice y and replace x by y.







DELETE 20 1 8



Binary Search Trees

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- Time. O(h)

Dynamic Ordered Sets

Data structure	SEARCH	INSERT	DELETE	Space
linked list	O(n)	O(1)	O(1)	O(n)
sorted array	O(log n)	O(n)	O(n)	O(n)
binary search tree	O(h)	O(h)	O(h)	O(n)

- Height. Depends on sequence of operations.
  - $h = \Omega(n)$ worst-case and $h = \Theta(\log n)$ on average.
- Challenge. Can we maintain height at $O(\log n)$ worst-case?

Search Trees

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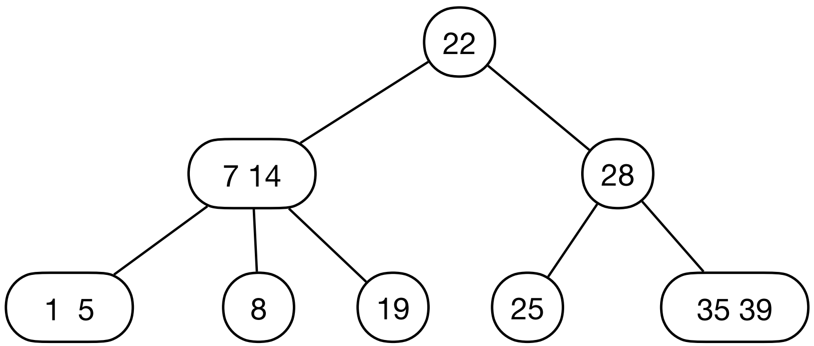
- Balanced Search Trees

Balanced Search Trees

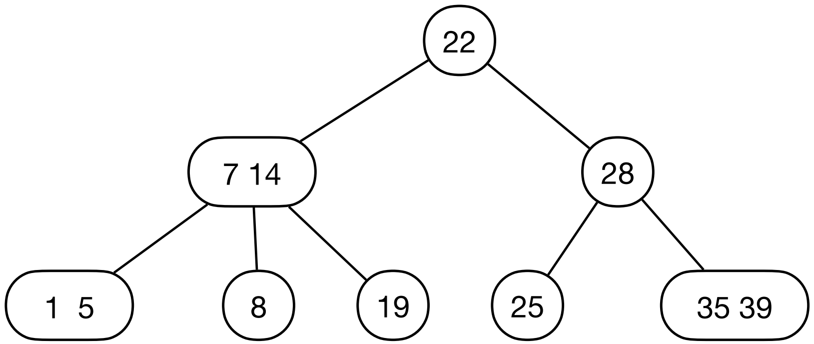
- 2-3 Tree.
  - Rooted tree.
  - Each internal node has 2 or 3 children.
    - 2-node: 2 children and 1 key
    - 3-node: 3 children and 2 keys.
  - Symmetric order.
    - Inorder traversal outputs the keys in sorted order.
  - Perfect balance.
    - Every path from root to a leaf has the same length
  - ⇒ height of tree is $\Theta(\log n)$



Balanced Search Trees

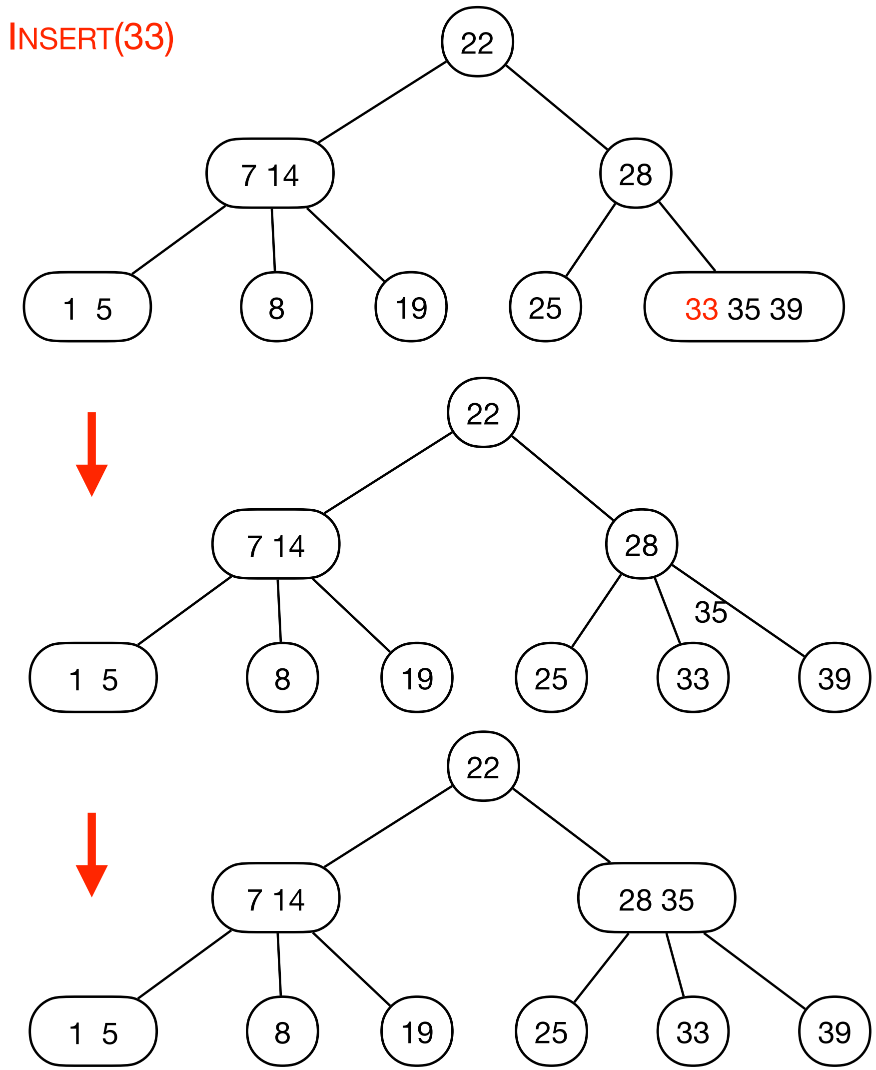
- SEARCH(k): traverse tree top-down.
  - Compare key k against keys in node.
  - If equal return element. Otherwise, recurse in child with interval containing k and recurse.
  - If we reach bottom, return null.

- Time. O(log n)

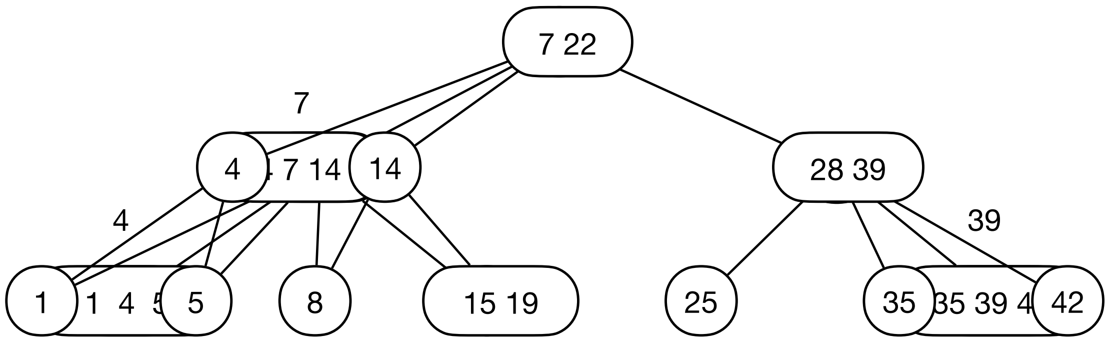


Balanced Search Trees

- INSERT(x):
  - Search for x.
  - Add x at leaf.
  - If too large, move middle key to parent. Repeat if necessary.
- Time. O(log n)



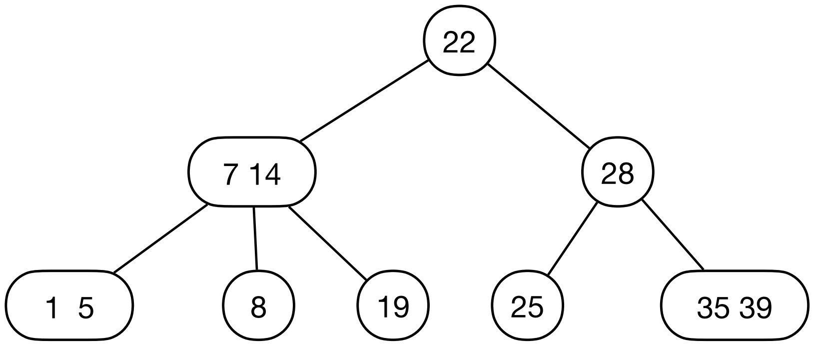
INSERT 15 42 4



Balanced Search Trees

- DELETE(x):
  - Search for x.
  - If x is a not a leaf, find node with smallest key > x.key, swap with x, and delete it.
  - If too small, take from parent. Repeat if necessary.

- Time. O(log n)



Dynamic Ordered Sets

Data structure	SEARCH	INSERT	DELETE	Space
linked list	O(n)	O(1)	O(1)	O(n)
sorted array	O(log n)	O(n)	O(n)	O(n)
binary search tree	O(h)	O(h)	O(h)	O(n)
2-3 tree	O(log n)	O(log n)	O(log n)	O(n)

- Can we do better?
  - Many variants of balanced search trees supporting many different operations.
  - Many efficient practical solutions.
  - Optimal time bounds for comparison-based data structures.
  - Even better bounds possible with more advanced techniques.

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