Dynamic Range Minimum Queries

Inge Li Gørtz

Range Minimum Queries

- Range minimum query problem. Preprocess array A[1…n] of integers to support
  - RMQ(i,j): return the (entry of) minimum element in A[i…j].

0	1	2	3	4	5	6	7	8	9
1	7	12	8	2	5	1	4	8	3

  - RMQ(2,5) = ?

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  - RMQ(2,5) = 2

- Solution 1. Store the array. Given a query run through array.
  - Space O(n)
  - Time O(n)

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  - RMQ(2,5) = 2

- Solution 2. Store a matrix with answer to all possible queries.
  - Space $O(n^2)$
  - Time O(1)

	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1		7	7	7	2	2	1	1	1	1
2			12	8	2	2	1	1	1	1
3				8	2	2	1	1	1	1
4					2	2	1	1	1	1
5						5	1	1	1	1
6							1	1	1	1
7								4	4	3
8									8	3
9										3

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Add(6,2) →

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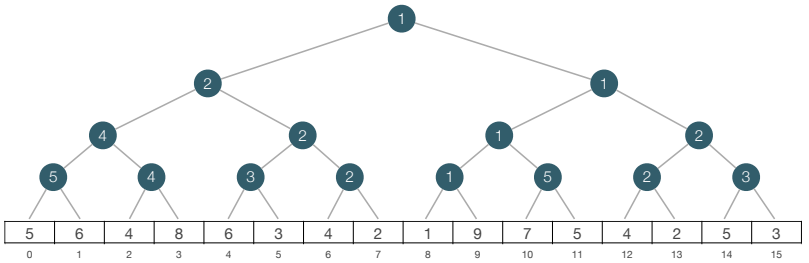

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- Dynamic RMQ: Support following operations.
  - Add(i, k): Set A[i] = A[i] + k   (k can be negative).
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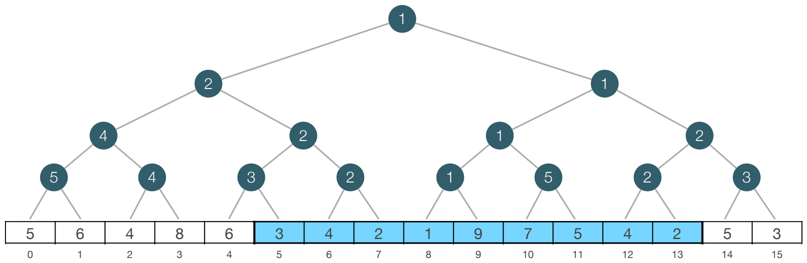
- RMQ(5,13) = ?



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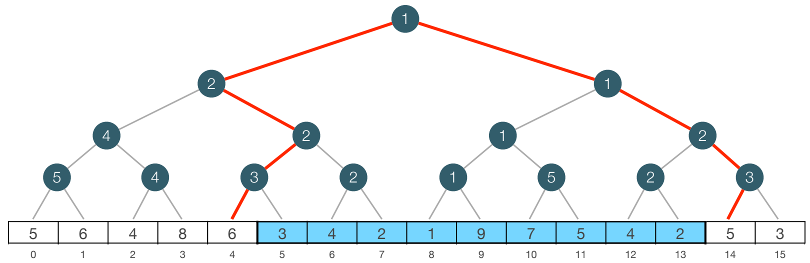

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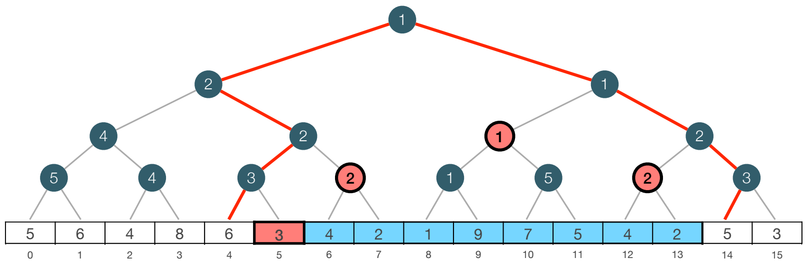

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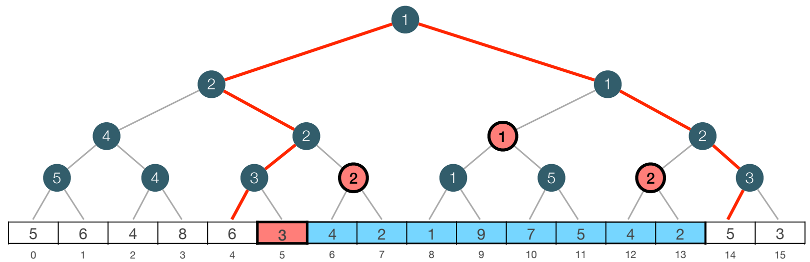

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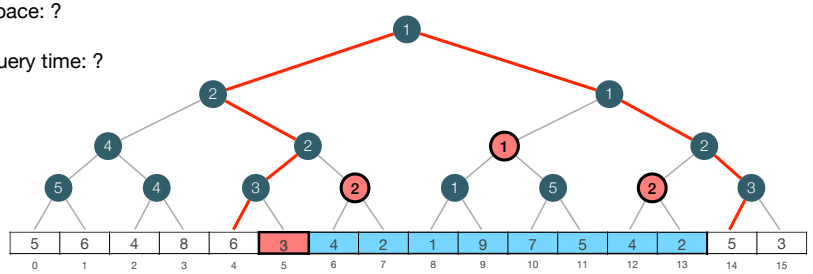
- Space: ?
- Query time: ?



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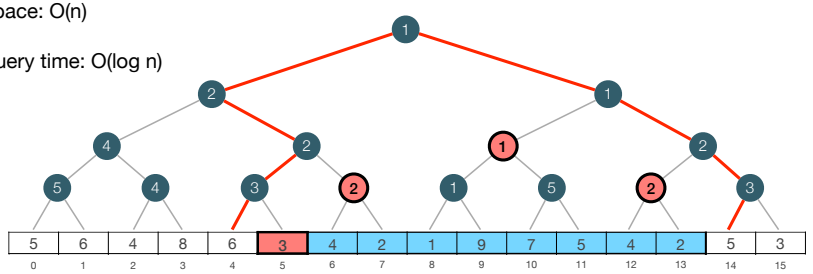
- Space: O(n)
- Query time: O(log n)



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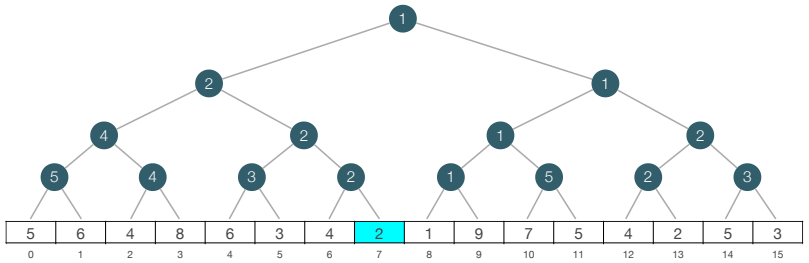

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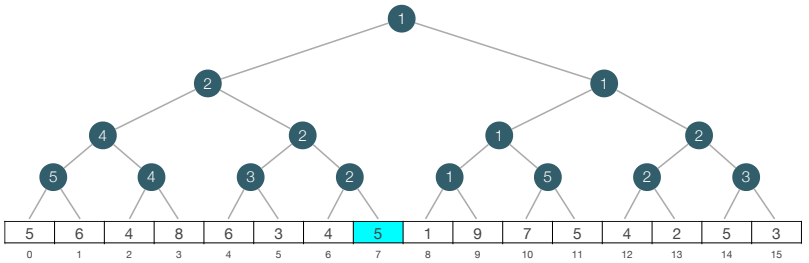

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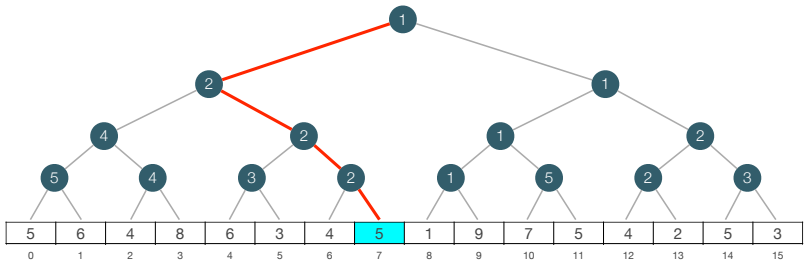

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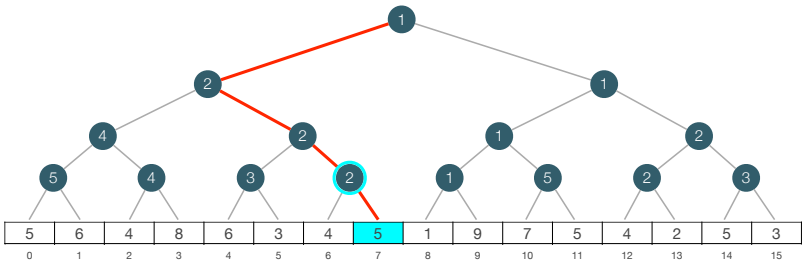

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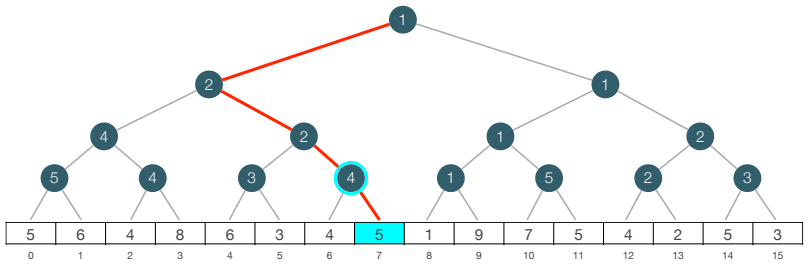

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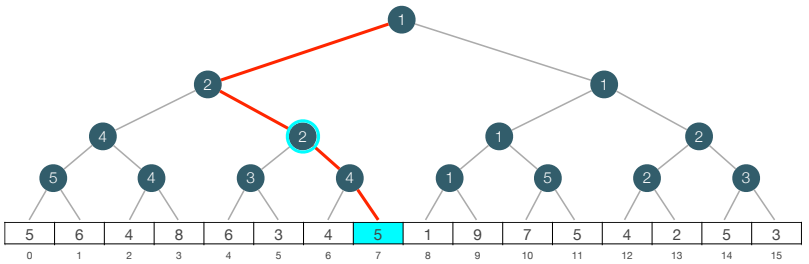

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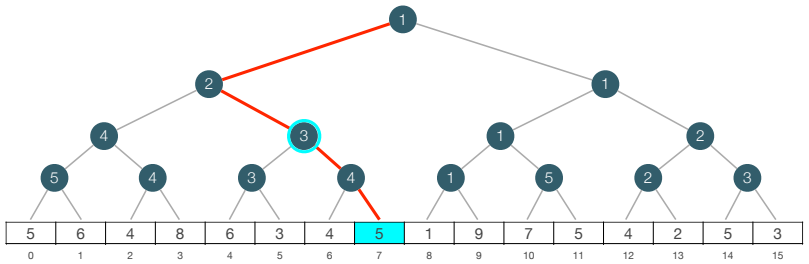

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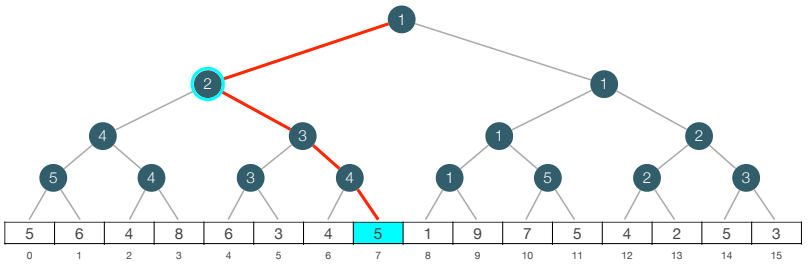

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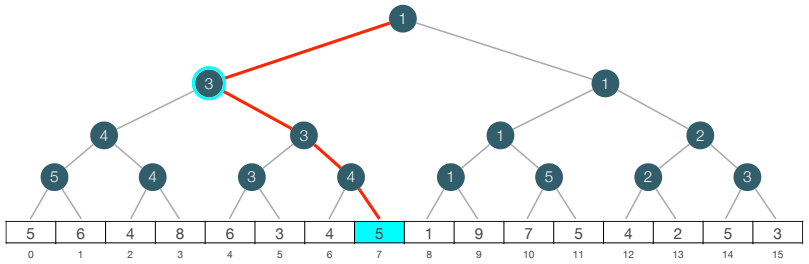

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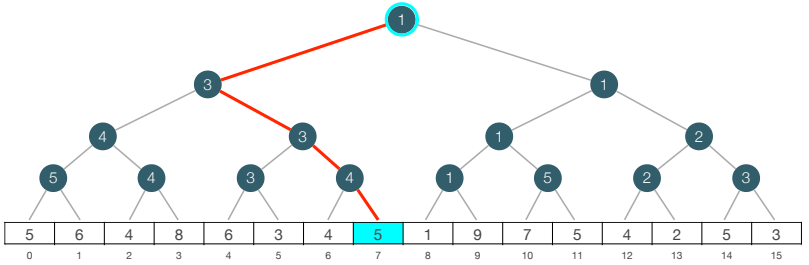

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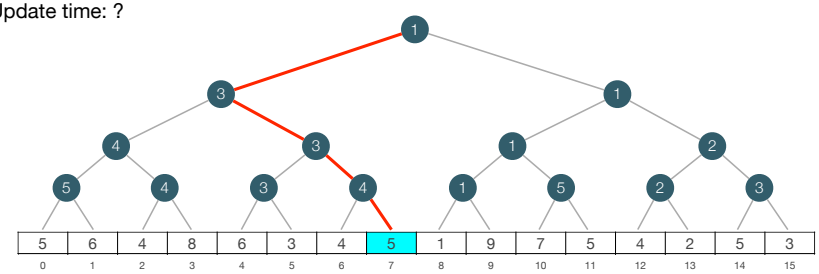

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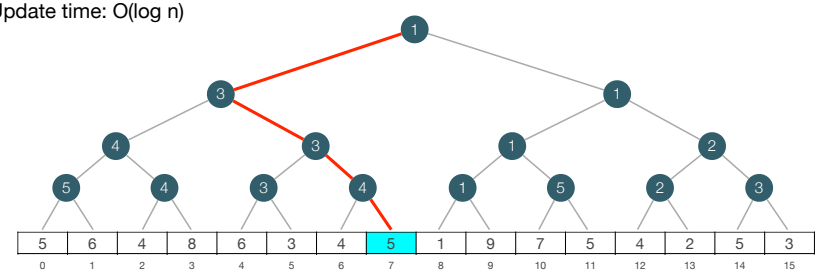

Compact Implementation

Segment trees

- Dynamic RMQ: Support following operations.
  - Add(i, k): Set A[i] = A[i] + k (k can be negative).
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s = INF
a = i, b = j
while (a not right of b):
  if (a right child):
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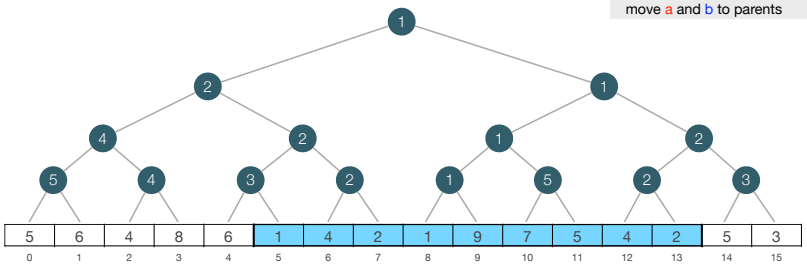

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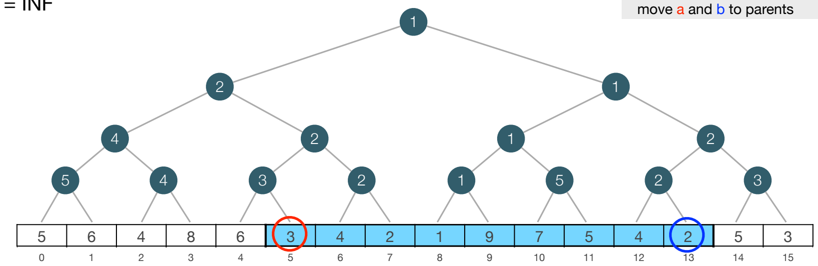

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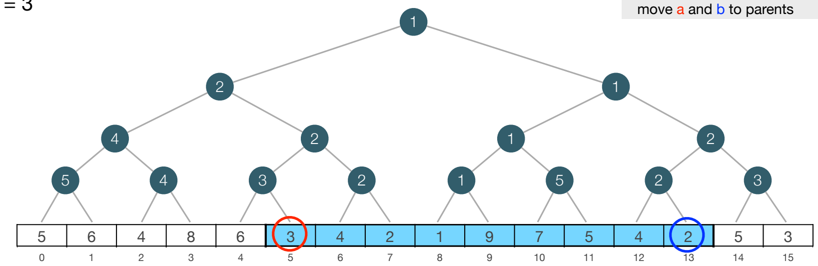

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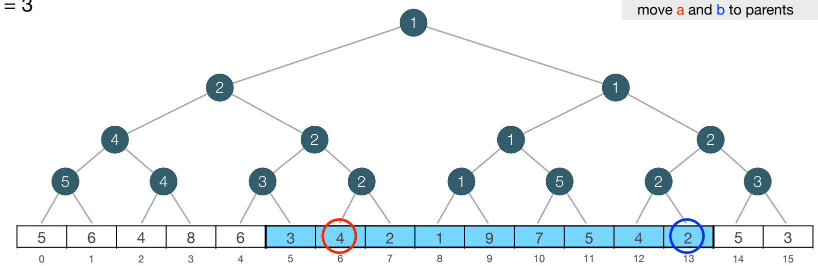

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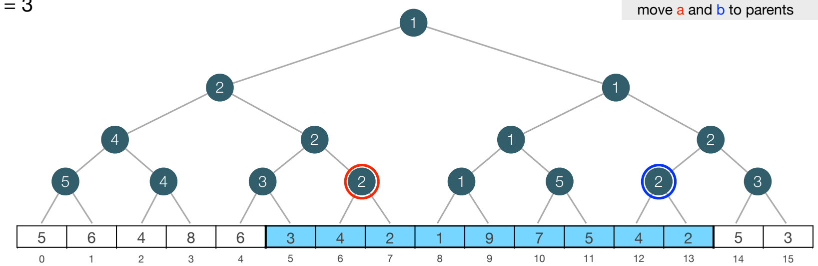

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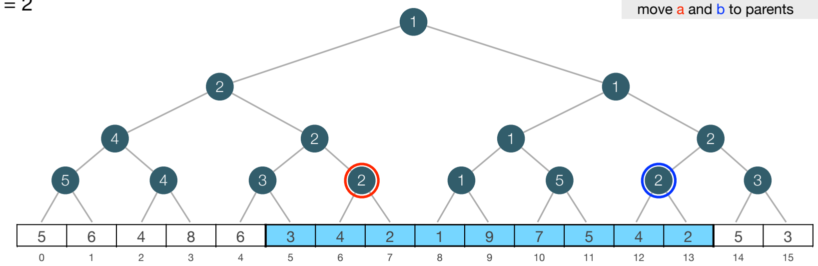

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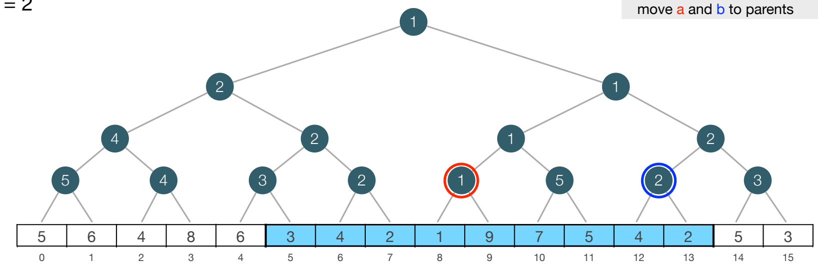

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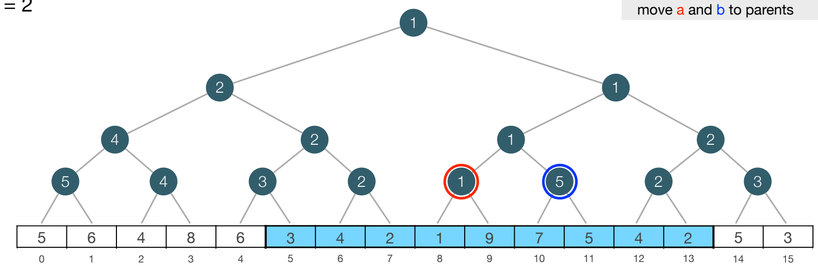

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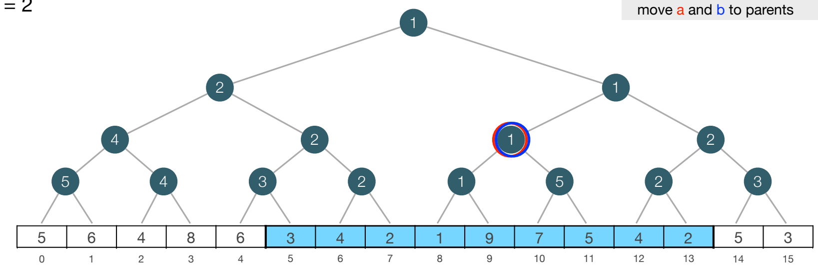

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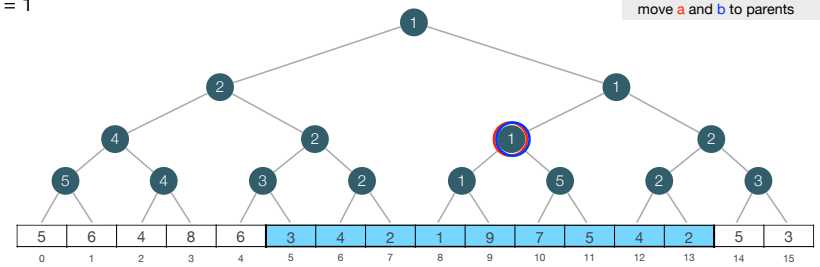

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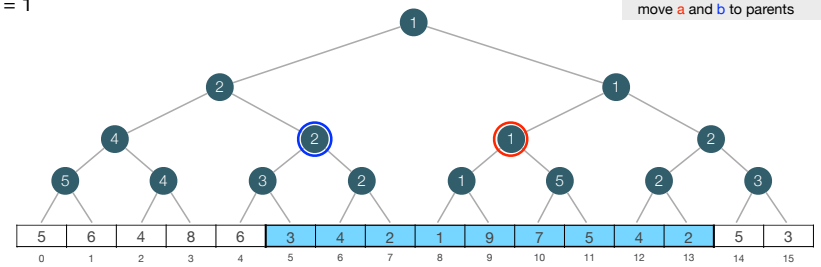

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  - RMQ(i,j)
- RMQ(5,13) = ?
- s = 1

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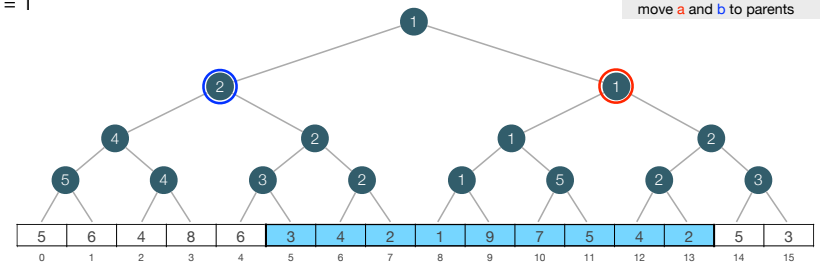

Segment trees

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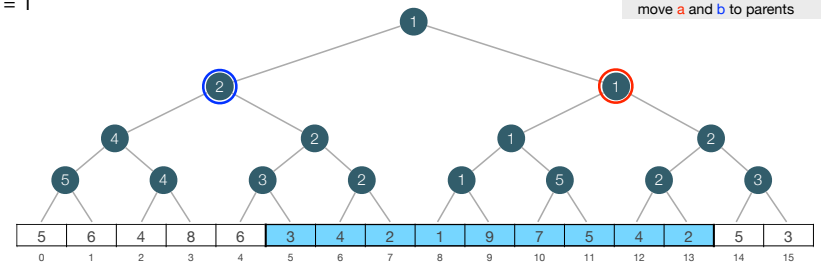
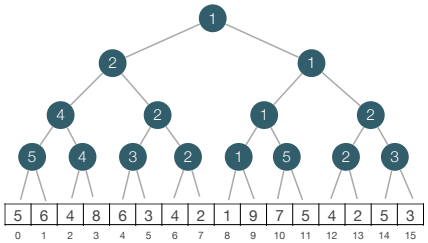


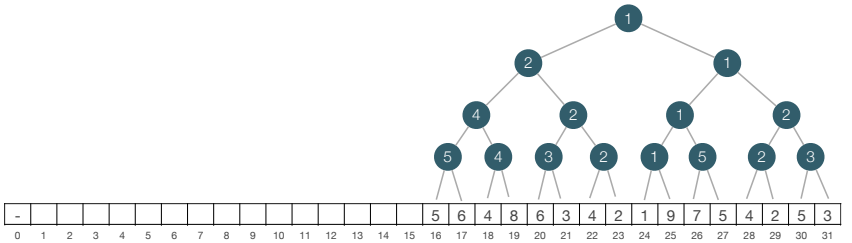
Segment trees

- Layout of tree:



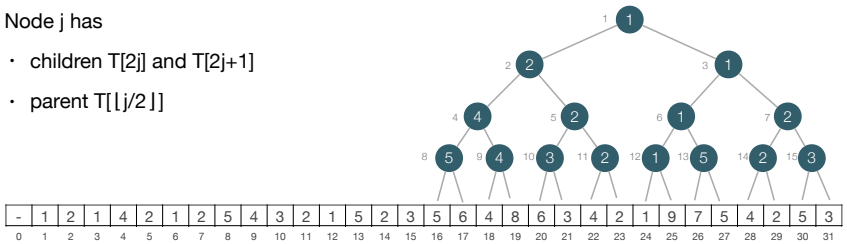
Segment trees

- Layout of tree:
  - Array T of length 2n.
  - T[n+i] = A[i]

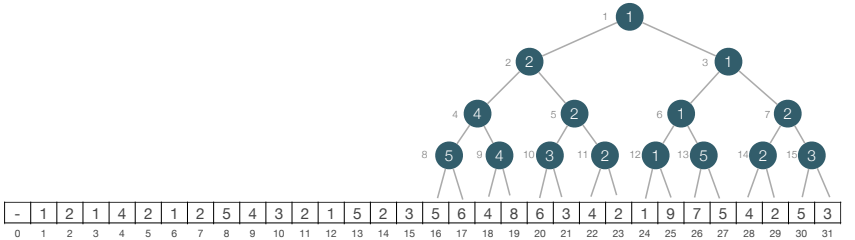


Segment trees

- Layout of tree: heap layout
  - Array T of length 2n.
  - T[n+i] = A[i]
  - T[1] is the root
  - T[2] is the left child and T[3] is the right child of the root.
  - Node j has
    - children T[2j] and T[2j+1]
    - parent T[⌊j/2⌋]



Segment trees

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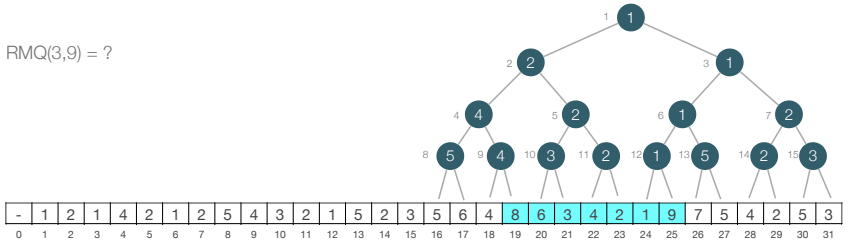
Segment trees

RMQ(3,9) = ?

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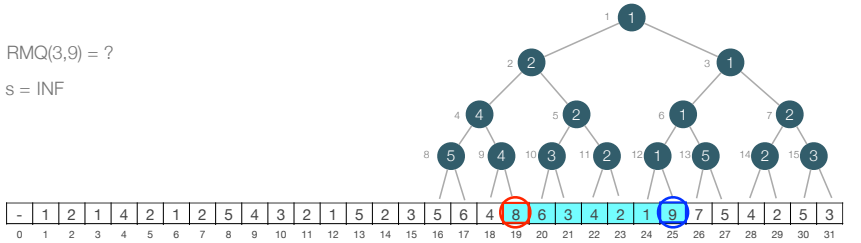


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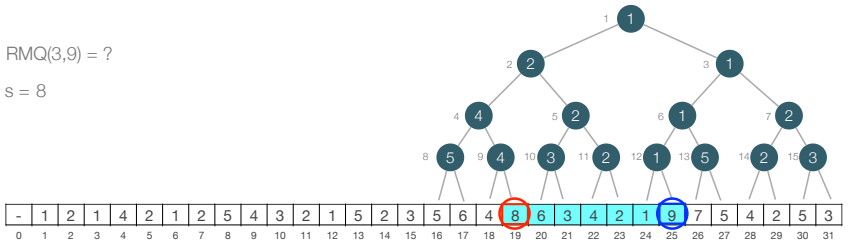
Segment trees

RMQ(3,9) = ?

s = 8

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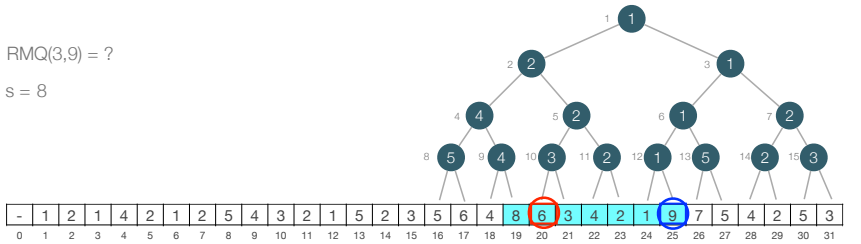


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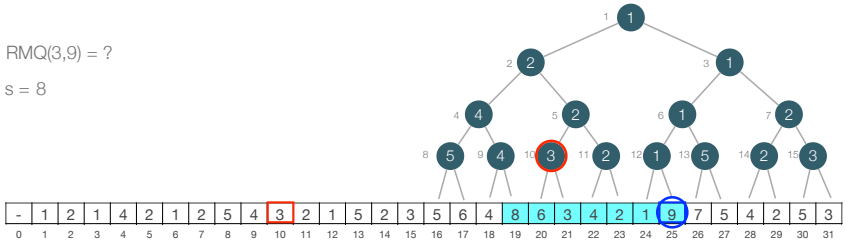


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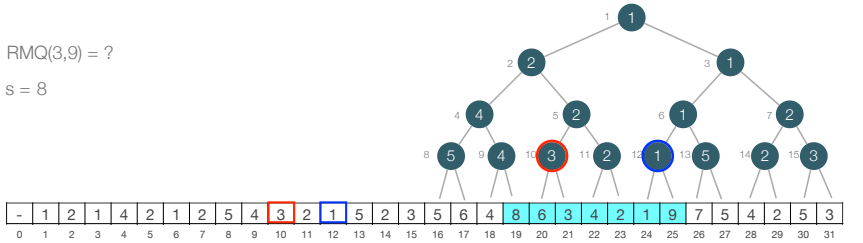


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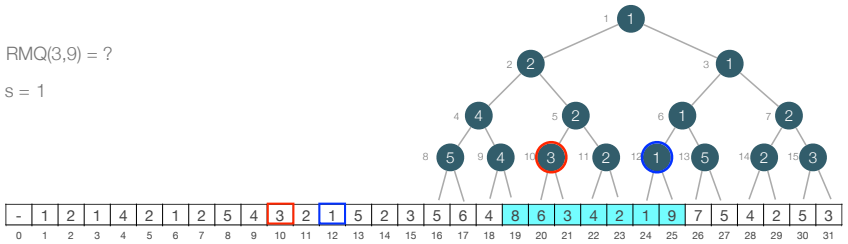


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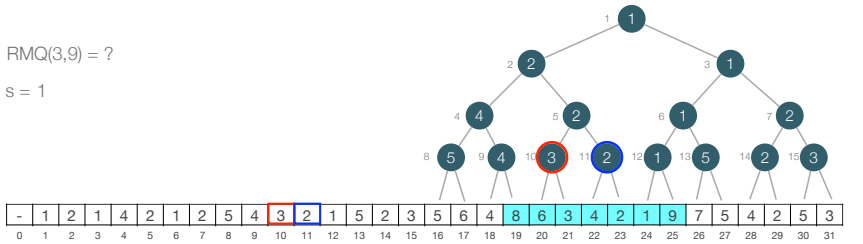


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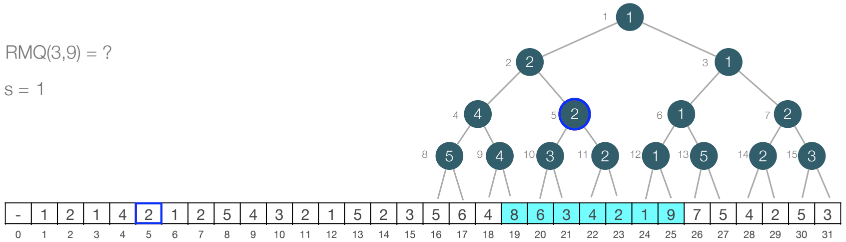


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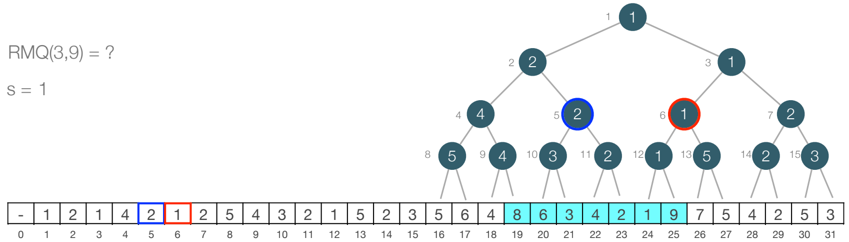


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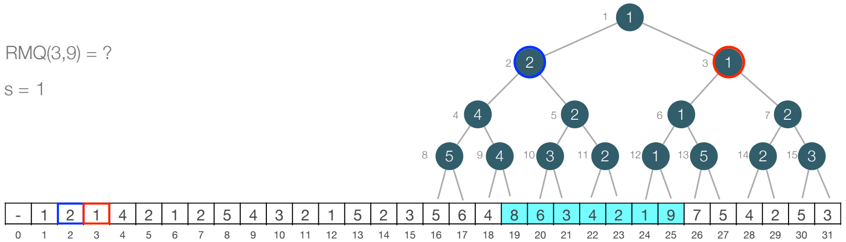


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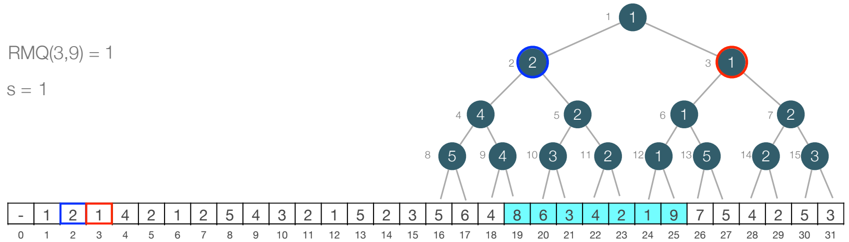


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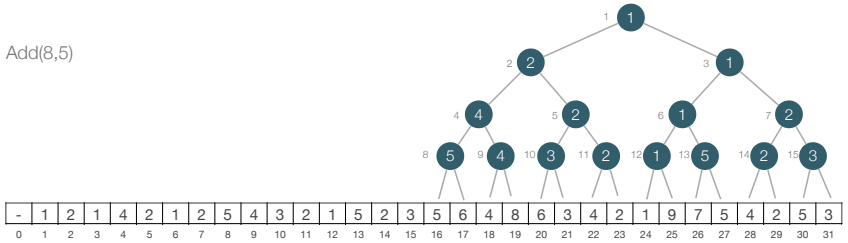
Segment trees

Add(8,5)

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  x = i + n
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  x = ⌊x/2⌋
  while (x ≥ 1):
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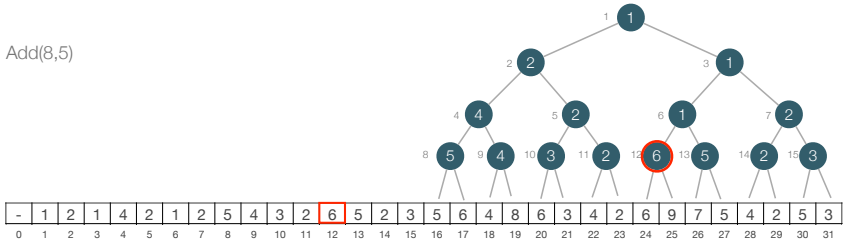


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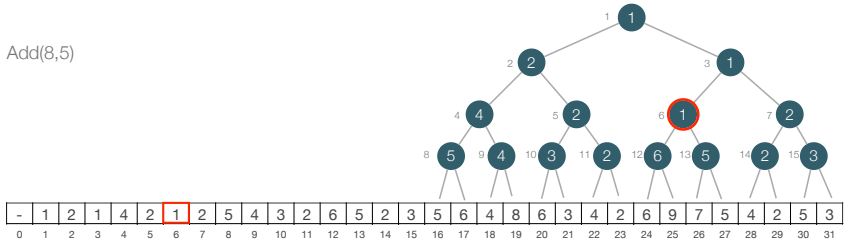


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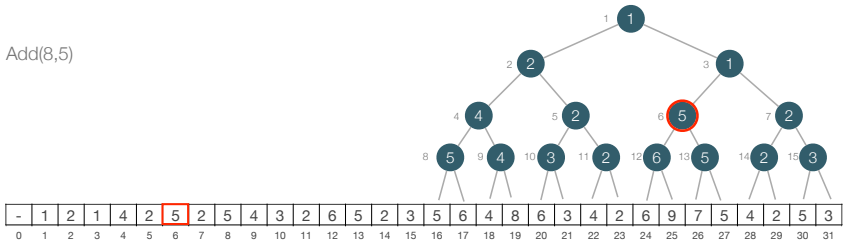


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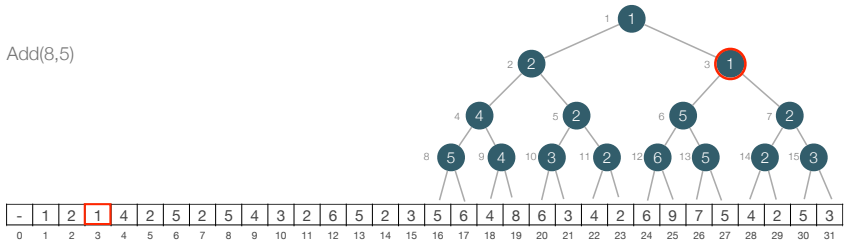


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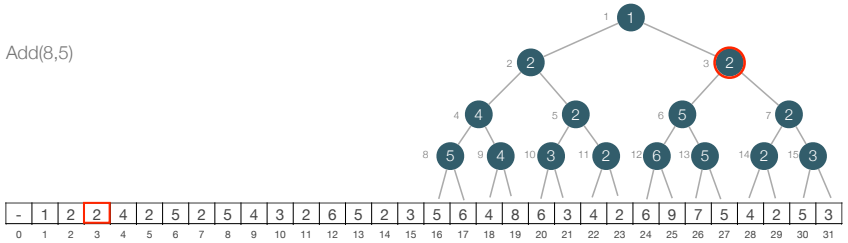


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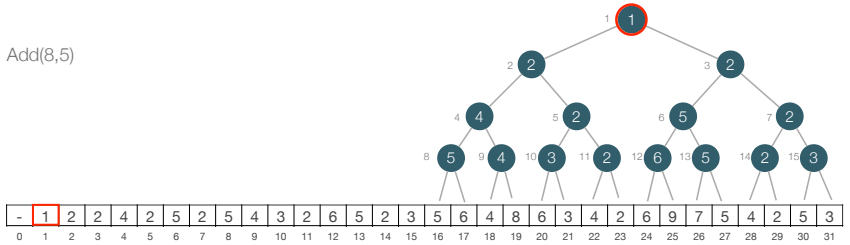


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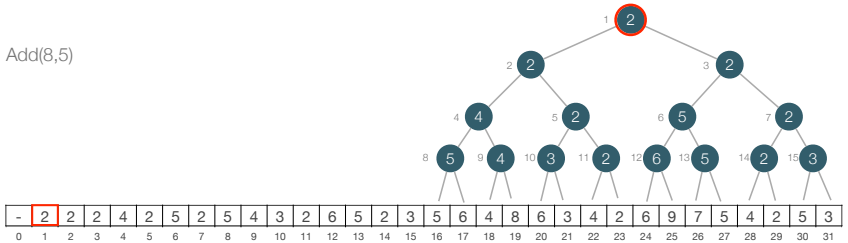


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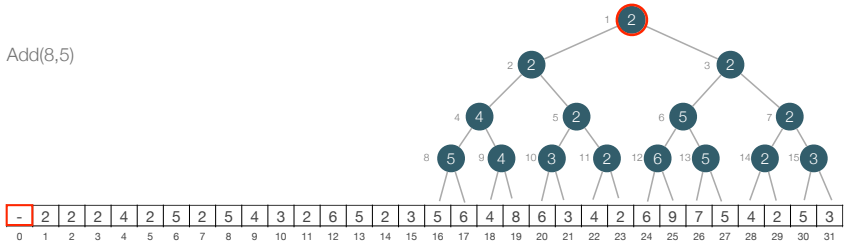
```

Segment trees

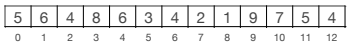
Add(8,5)




```
Add(i, k):
  x = i + n
  T[x] = T[x] +k
  x = ⌊x/2⌋
  while (x ≥ 1):
    T[x] = min(T[2x], T[2x+1])
    x = ⌊x/2⌋
```

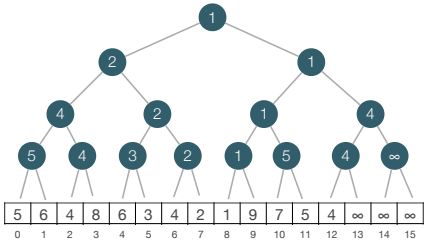
Segment tree

- What if n is not a power of two?



Segment tree

- What if n is not a power of two?



Segment tree

- What if n is not a power of two?