Partial Sums and Dynamic Arrays

- Partial Sums
- Dynamic Arrays

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Partial Sums

- Partial sums. Maintain array A[0,1,..., n] of integers support the following operations.
  - SUM(i): return A[1] + A[2] + ... + A[i]
  - UPDATE(i, Δ): set A[i] = A[i] + Δ

-	1	2	1	1	0	2	3	1	0	1	3	4	1	1	1	2
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Partial Sums

- Applications.
  - Dynamic lists and arrays (random access into changing lists)
  - Arithmetic coding.
  - Succinct data structures.
  - Lower bounds and cell probe complexity.
  - Basic component in many data structures.

- Challenge. How can solve the problem with current techniques?

Partial Sums

-	1	2	1	1	0	2	3	1	0	1	3	4	1	1	1	2
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

- Slow sum and ultra fast updates. Maintain A explicitly.
  - SUM(i): compute A[0] + .. + A[i].
  - UPDATE(i, Δ): set A[i] = A[i] + Δ
- Time.
  - O(i) = O(n) for SUM, O(1) for UPDATE.

Partial Sums

-	1	3	4	5	5	7	10	11	11	12	15	19	20	21	22	24
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

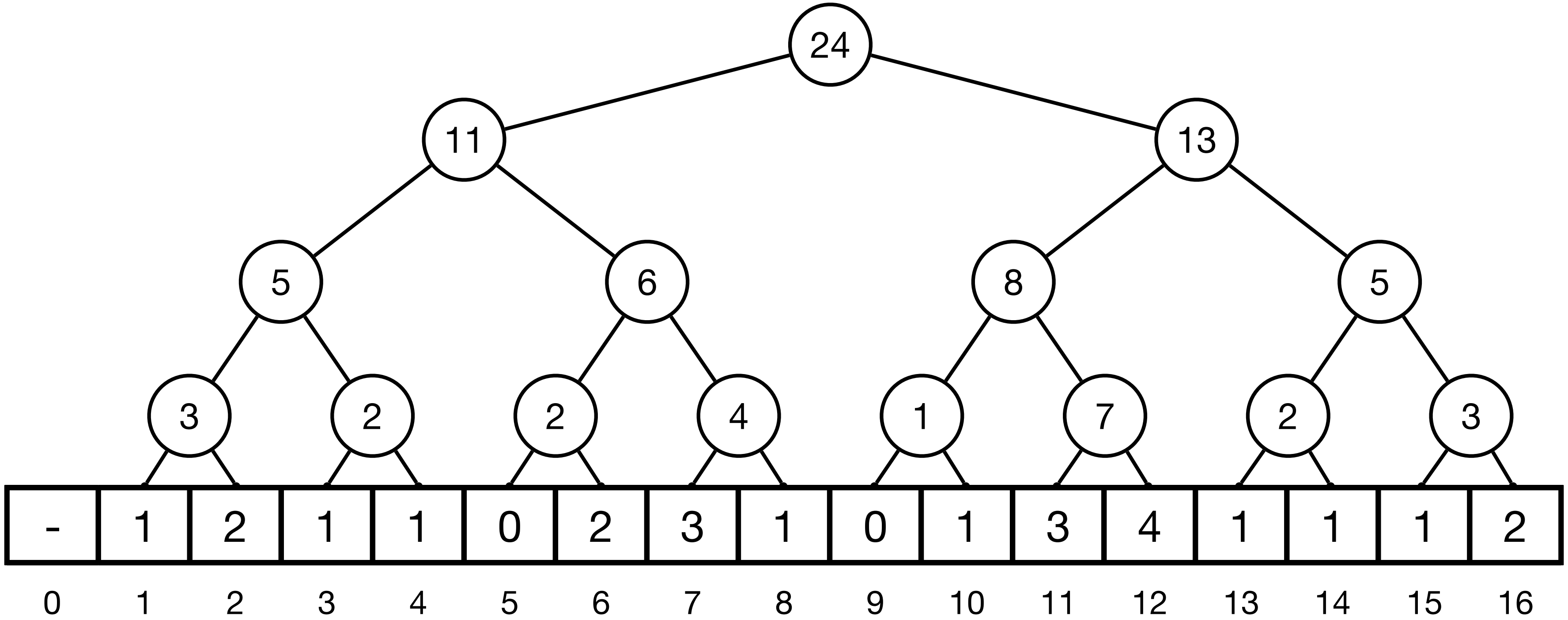
-	1	2	1	1	0	2	3	1	0	1	3	4	1	1	1	2
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

- Ultra fast sum and slow updates. Maintain partial sum P of A.
  - SUM(i): return P[i].
  - UPDATE(i, Δ): add Δ to P[i], P[i+1], ..., P[n].
- Time.
  - O(1) for SUM, O(n - i + 1) = O(n) for UPDATE.

Partial Sums

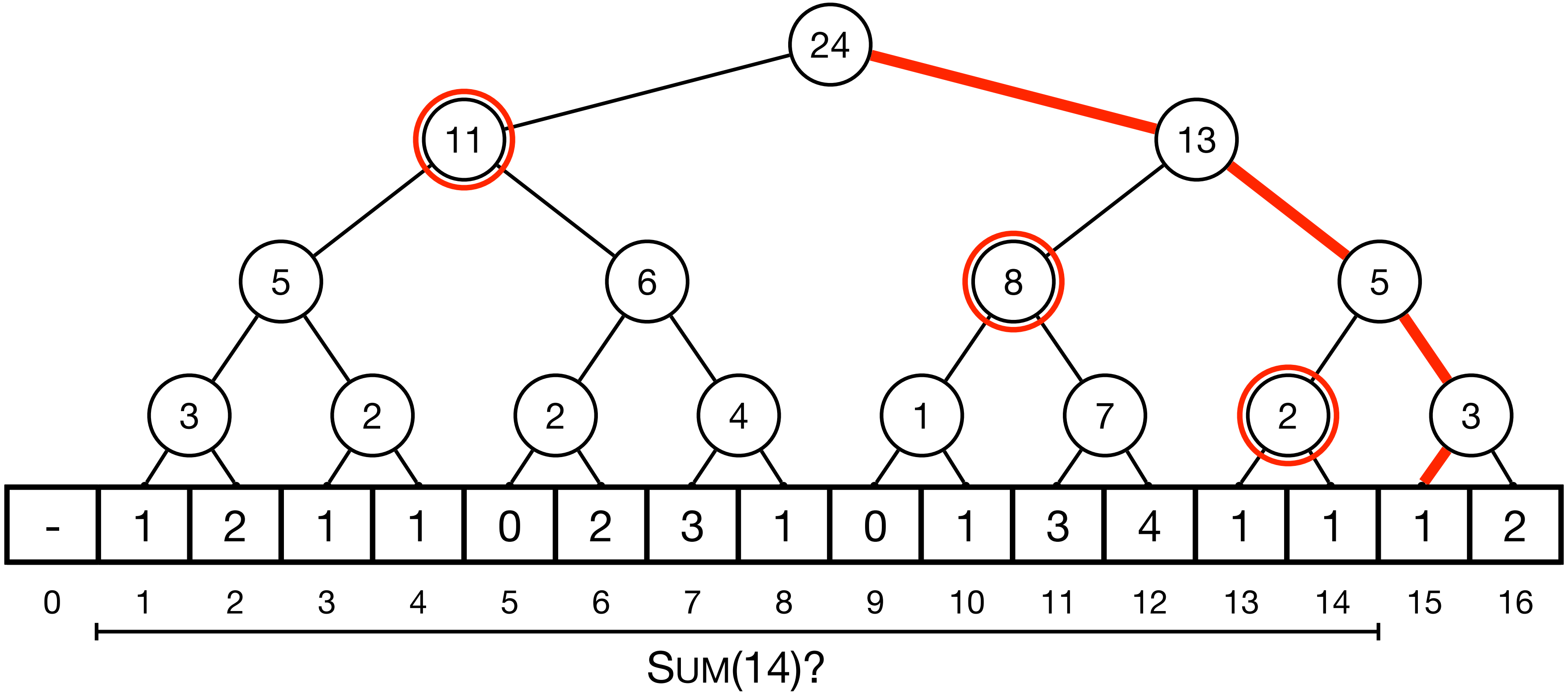
Data structure	SUM	UPDATE	Space
explicit array	O(n)	O(1)	O(n)
explicit partial sum	O(1)	O(n)	O(n)

Partial Sums



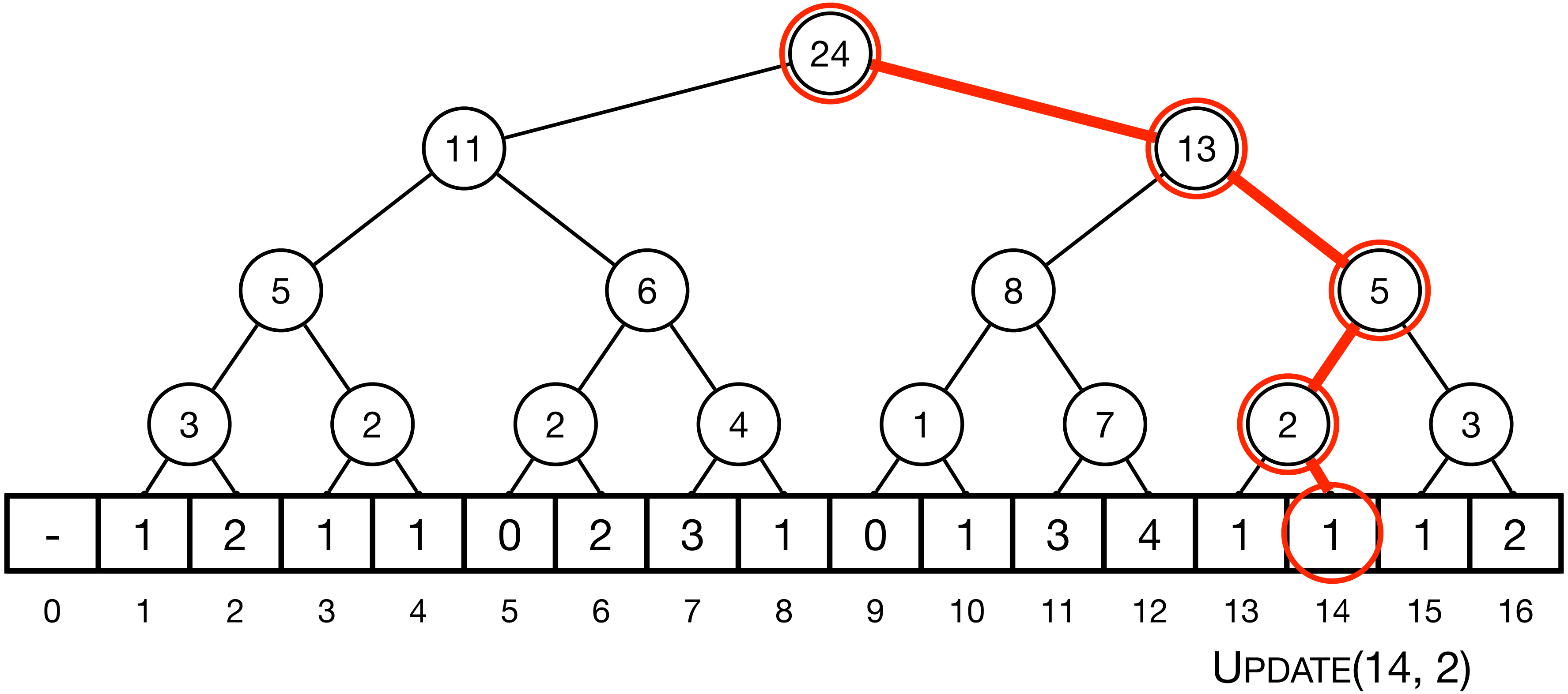
- Fast sum and fast updates. Maintain balanced binary tree T on A. Each node stores the sum of elements in subtree.

Partial Sums



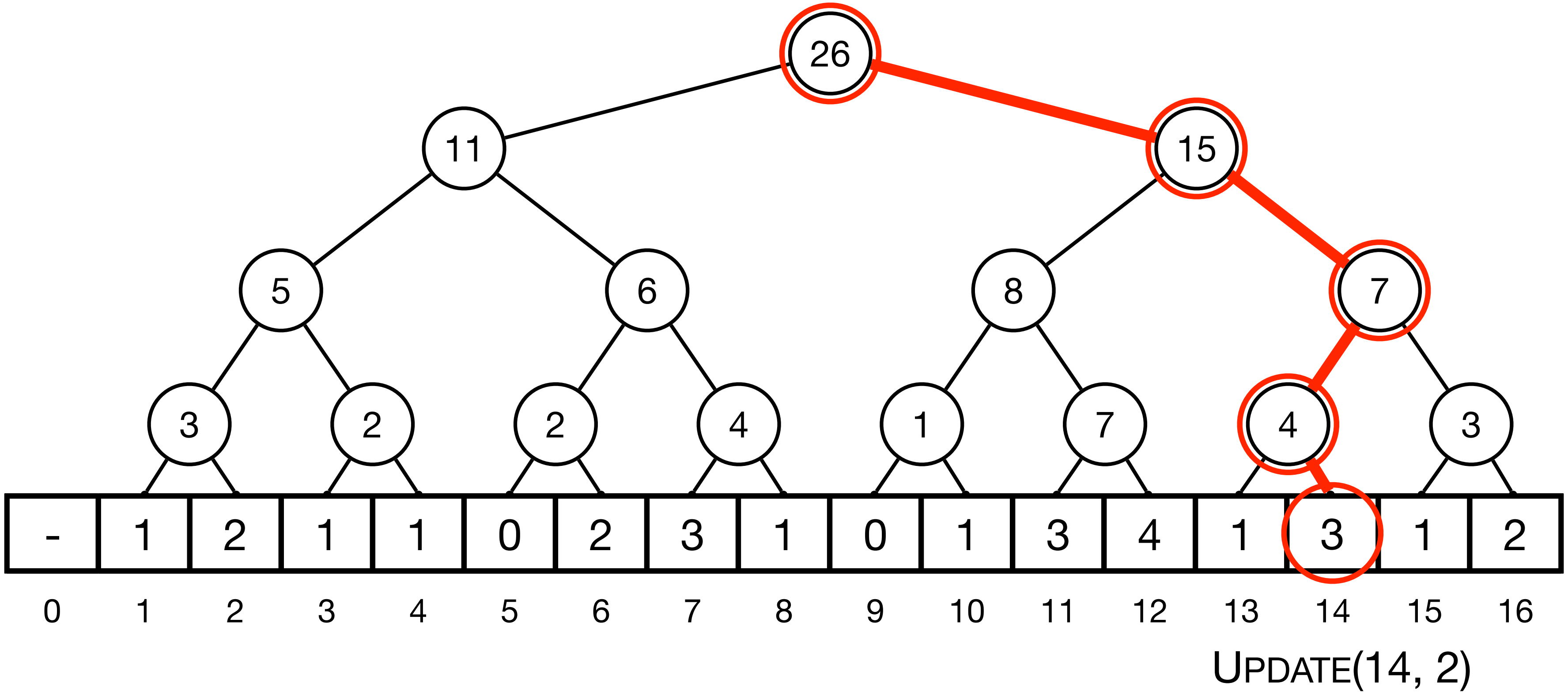
- SUM.
  - SUM(i): traverse path to i + 1 and sum up all off-path nodes.
- Time. O(log n)

Partial Sums



- UPDATE.
  - UPDATE(i, Δ): add Δ to nodes on path to i.

Partial Sums



- UPDATE.
  - UPDATE(i, Δ): add Δ to nodes on path to i.
- Time. O(log n)

Partial Sums

Data structure	SUM	UPDATE	Space
explicit array	O(n)	O(1)	O(n)
explicit partial sum	O(1)	O(n)	O(n)
balanced binary tree	O(log n)	O(log n)	O(n)
lower bound	Ω(log n)	Ω(log n)	

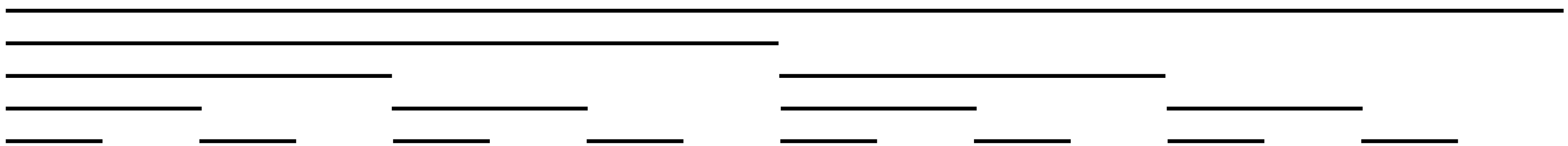
- Challenge. How can we improve?
- In-place data structure.
  - Replace input array A with data structure of exactly same size.
  - Use only O(1) extra space.

Partial Sums

-	1	3	1	5	0	2	3	11	0	1	3	8	1	2	1	24
-	1	3	1	5	0	2	3	11	0	1	3	8	1	2	1	13
-	1	3	1	5	0	2	3	6	0	1	3	8	1	2	1	5
-	1	3	1	2	0	2	3	4	0	1	3	7	1	2	1	3
-	1	2	1	1	0	2	3	1	0	1	3	4	1	1	1	2
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

- Fenwick tree. Replace A by another array F.
  - Replace all even entries A[2i] by A[2i - 1] + A[2i].
  - Recurse on the entries A[2, 4, .., n] until we are left with a single element.

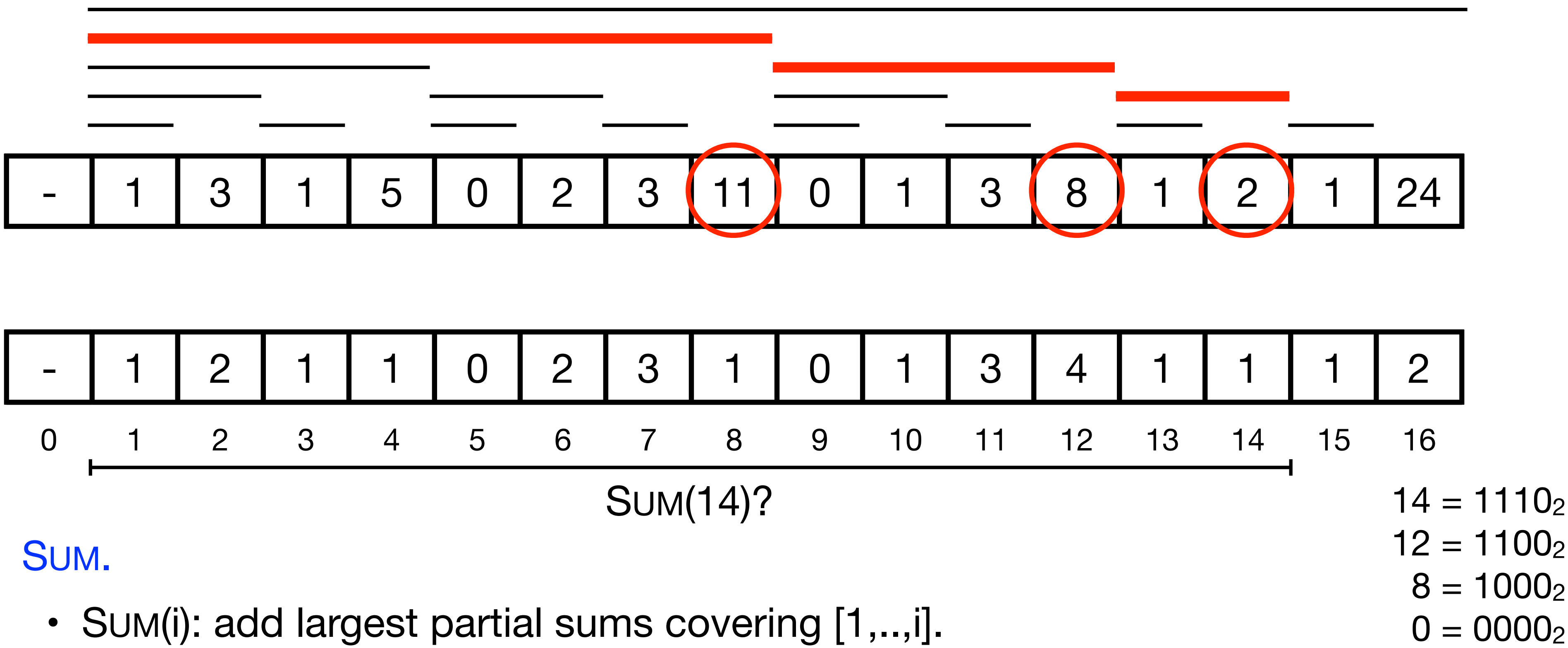
Partial Sums



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
-	1	3	1	5	0	2	3	11	0	1	3	8	1	2	1	24
-	1	2	1	1	0	2	3	1	0	1	3	4	1	1	1	2

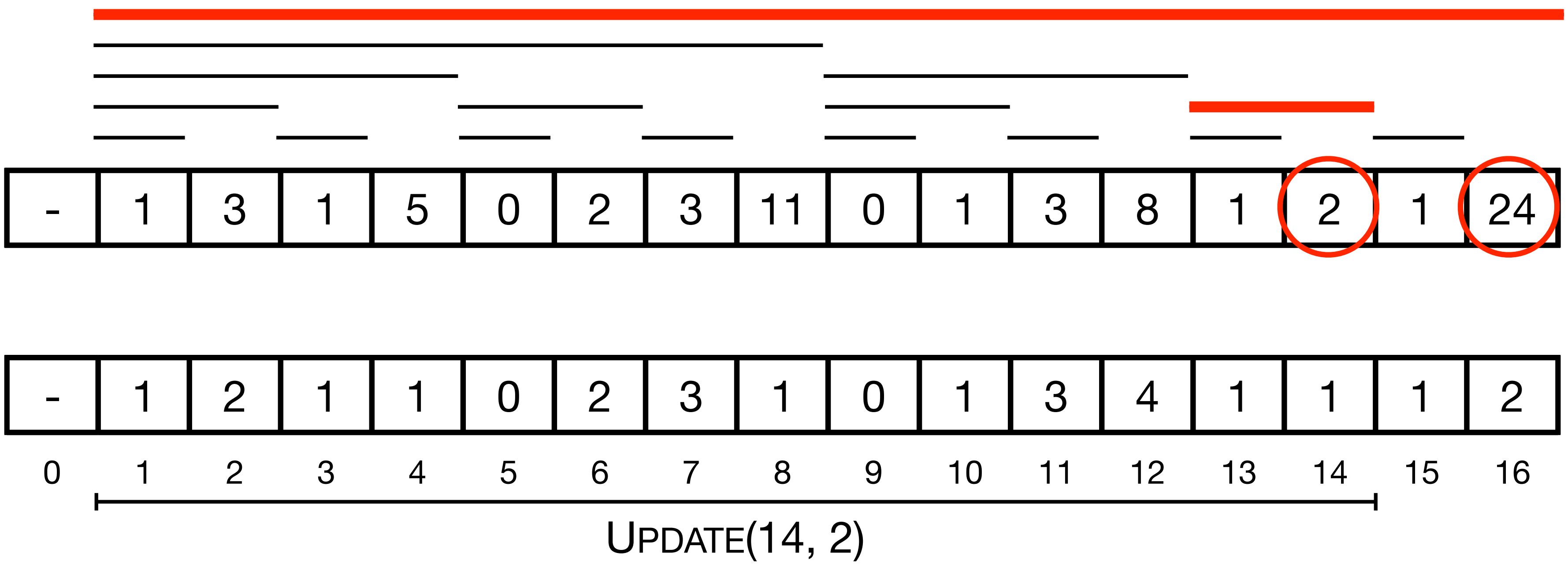
- Fenwick tree. Replace A by another array F.
  - Replace all even entries A[2i] by A[2i - 1] + A[2i].
  - Recurse on the entries A[2, 4, .., n] until we are left with a single element.
- Space.
  - In-place. No extra space.

Partial Sums



- SUM.
  - SUM(i): add largest partial sums covering [1,..,i].
  - Indexes i_0, i_1, .. in F given by $i_0 = i$ and $i_{j+1} = i_j - rmb(i_j)$, where $rmb(i_j)$ is the integer corresponding to the rightmost 1-bit in i. Stop when we get 0.
- Time. O(log n)

Partial Sums



14 = 1110_2
16 = 10000_2

- UPDATE.
  - UPDATE(i, Δ): add Δ to partial sums covering i.
  - Indexes i_0, i_1, .. in F given by $i_0 = i$ and $i_{j+1} = i_j + rmb(i_j)$. Stop when we get n.
- Time. O(log n)

Partial Sums

Data structure	SUM	UPDATE	Space
explicit array	O(n)	O(1)	O(n)
explicit partial sum	O(1)	O(n)	O(n)
balanced binary tree	O(log n)	O(log n)	O(n)
lower bound	Ω(log n)	Ω(log n)	
Fenwick tree	O(log n)	O(log n)	in-place

- Practical? Fenwick trees for competitive programming.

Partial Sums and Dynamic Arrays

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Dynamic Arrays

- Dynamic arrays. Maintain array A[0,..., n-1] of integers support the following operations.
  - ACCESS(i): return A[i].
  - INSERT(i, x): insert a new entry with value x immediately to the left of entry i.
  - DELETE(i): Remove entry i.

1	2	1	1	0	2	3	1	0	1	3	4	1	1	1	2
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Dynamic Arrays

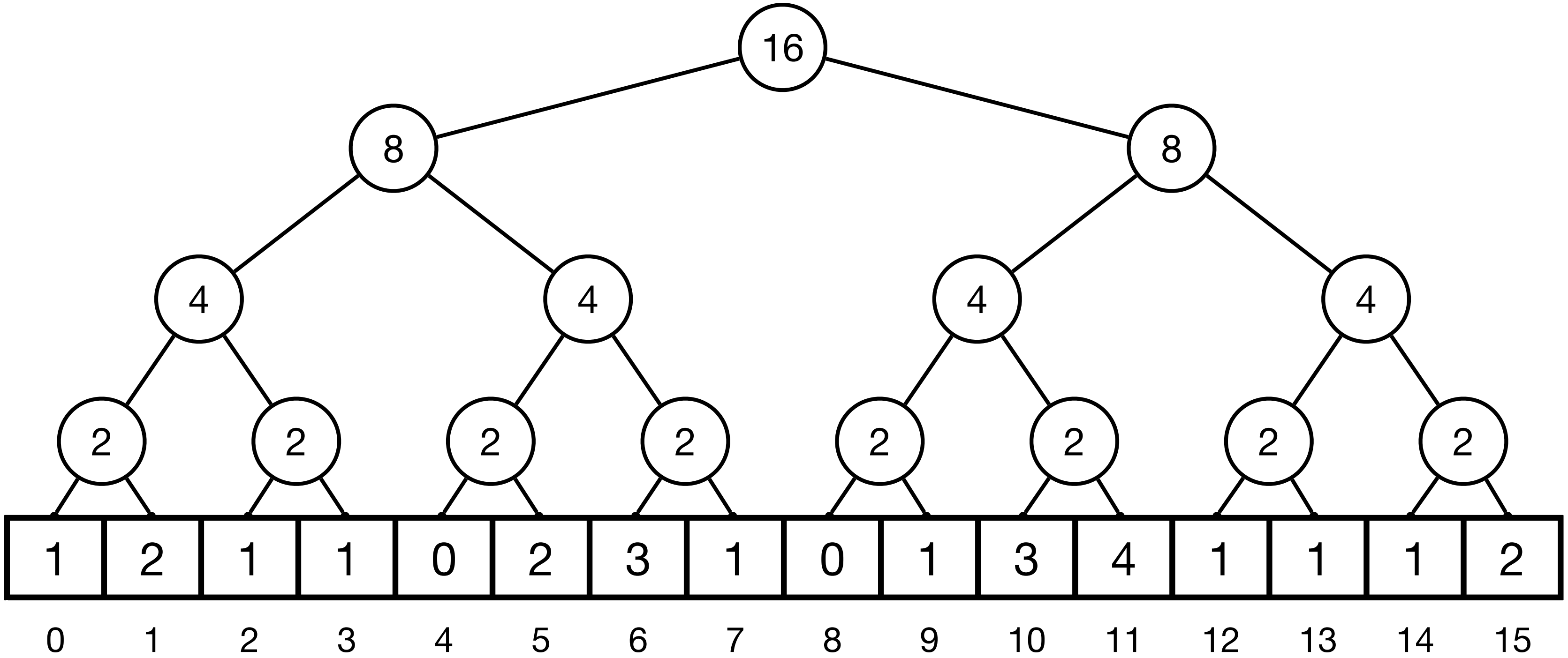
- Applications.
  - Dynamic lists and arrays (random access into changing lists)
  - Basic component in many data structures.
- Challenge. How can solve the problem with current techniques?

Dynamic Arrays

1	2	1	1	0	2	3	1	0	1	3	4	1	1	1	2
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

- Very fast access and slow updates. Maintain A explicitly.
  - ACCESS(i): return A[i].
  - INSERT(i, x): Shift all elements from i to n by 1 to the right. Set A[i] = x.
  - DELETE(i): shift all elements to the right of entry i to the left by 1.
- Time.
  - O(1) for ACCESS and O(n-i+1) = O(n) for INSERT and DELETE.

Dynamic Arrays



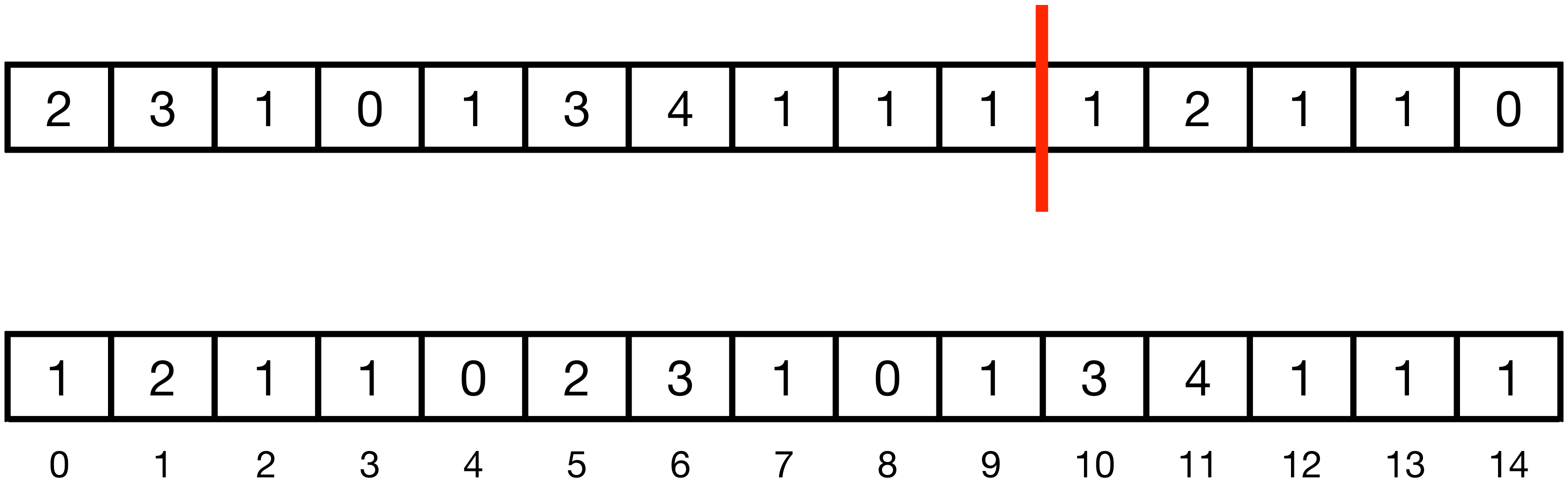
- Fast access and fast updates. Maintain balanced binary tree T on A. Each node stores the number of elements in subtree.
  - ACCESS(i): traverse path to leaf j.
  - INSERT(i, x): insert new leaf and update tree.
  - DELETE(i): delete new leaf and update tree.
- Time. O(log n) for ACCESS, INSERT, and DELETE.

Dynamic Arrays

Data structure	ACCESS	INSERT	DELETE	Space
explicit array	O(1)	O(n)	O(n)	O(n)
balanced binary tree	O(log n)	O(log n)	O(log n)	O(n)
lower bound	Ω(log n/log log n)	Ω(log n/log log n)	Ω(log n/log log n)	

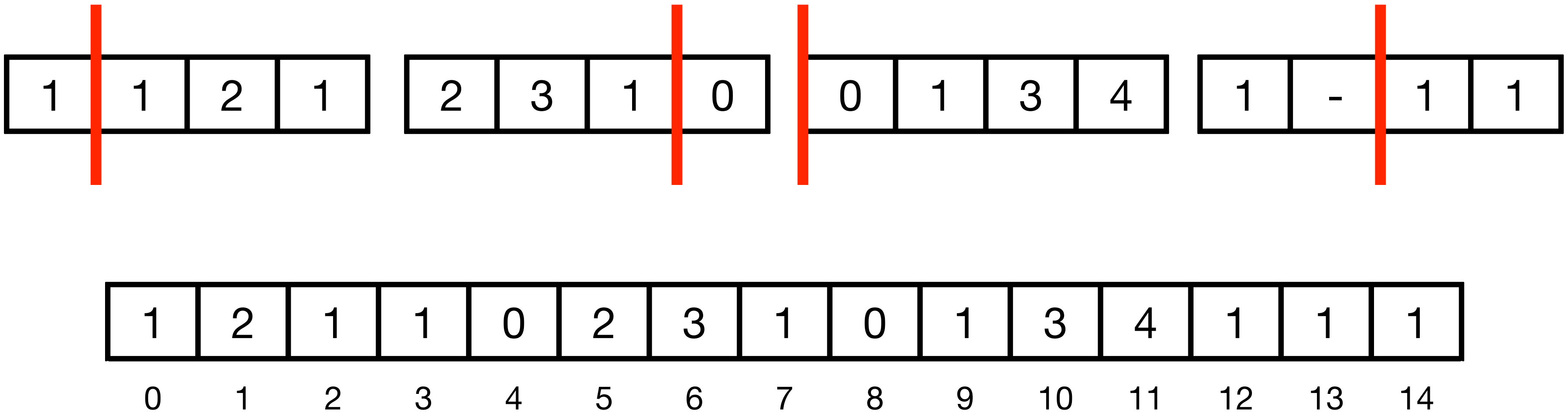
- Challenge. What can we get if we insist on constant time ACCESS?

Dynamic Arrays



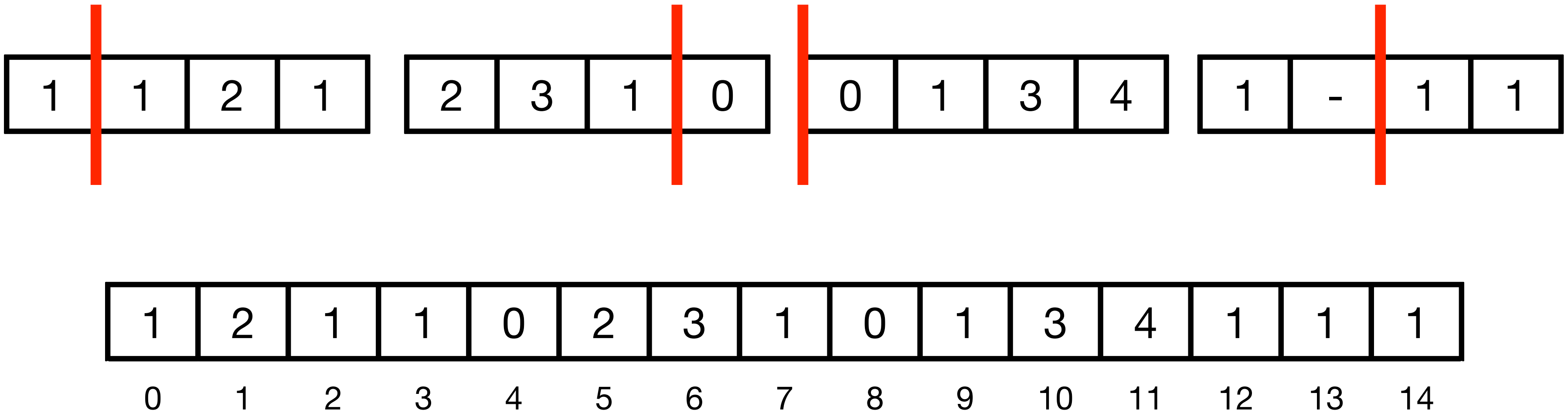
- Rotated array.
  - Circular shift of array by an offset.
- Idea.
  - By moving offset we can delete and insert at endpoints in O(1) time.
  - Lead to underflow or overflow.

Dynamic Arrays



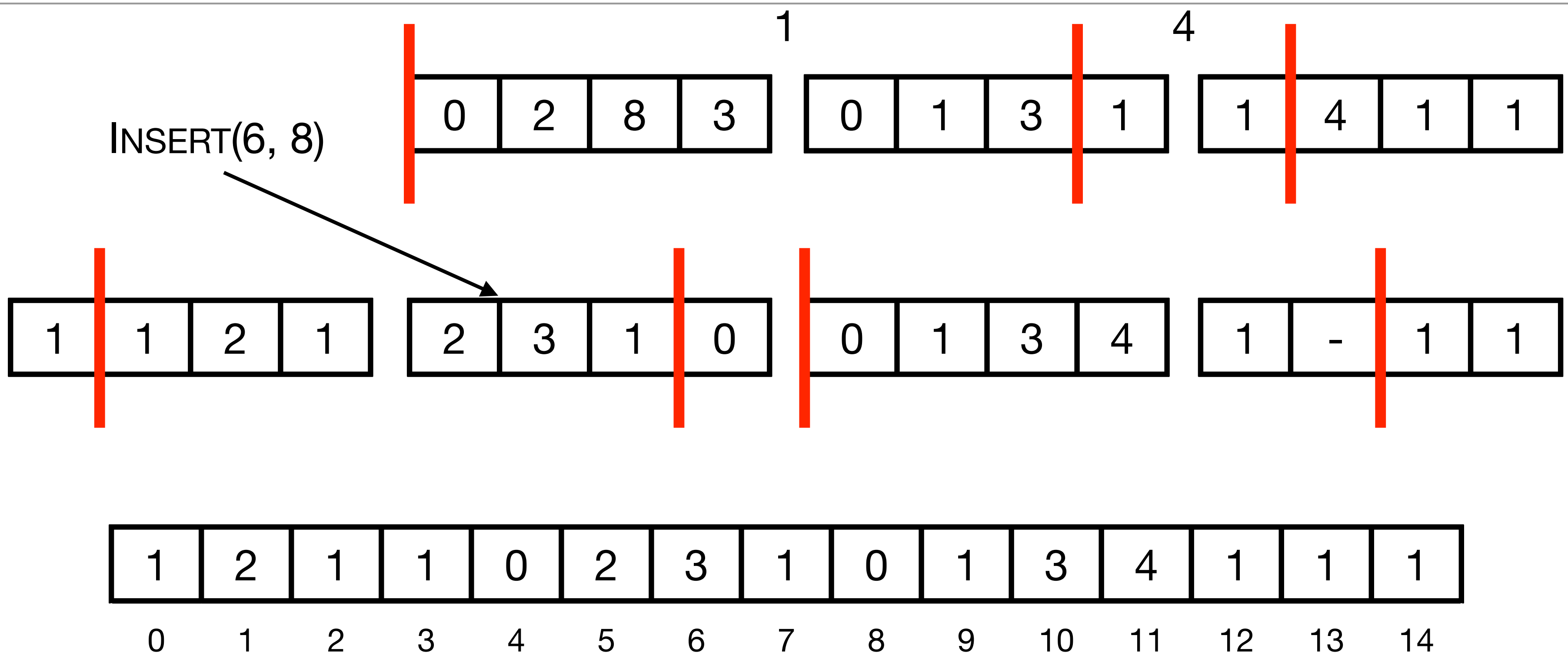
- 2-level rotated arrays.
  - Store $\sqrt{n}$ rotated arrays $R_0, \ldots, R_{\sqrt{n}-1}$ with capacity $\sqrt{n}$ (last may have smaller capacity).

Dynamic Arrays



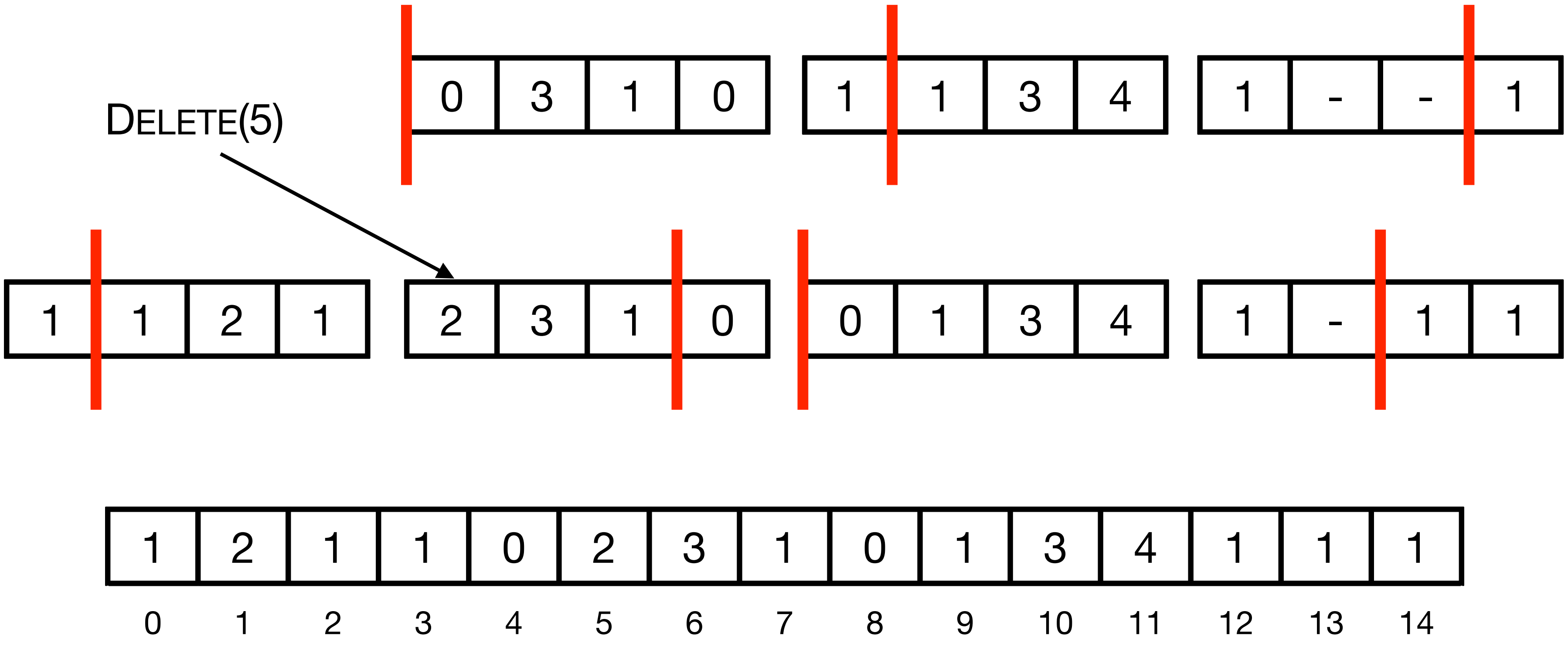
- ACCESS.
  - ACCESS(i): compute rotated array R_j and index k corresponding to i. Return $R_j[k]$.
- Time. O(1)

Dynamic Arrays



- INSERT.
  - INSERT(i, x): find R_j and k as in ACCESS.
    - Rebuild R_j with new entry inserted.
    - Propagate overflow to R_{j+1} recursively.
- Time. $O(\sqrt{n})$

Dynamic Arrays



- DELETE.
  - DELETE(i): find R_j and k as in ACCESS.
    - Rebuild R_j with entry i deleted.
    - Propagate underflow to R_{j+1} recursively.
- Time. $O(\sqrt{n})$

Dynamic Arrays

Data structure	ACCESS	INSERT	DELETE	Space
explicit array	O(1)	O(n)	O(n)	O(n)
balanced binary tree	O(log n)	O(log n)	O(log n)	O(n)
lower bound	Ω(log n/log log n)	Ω(log n/log log n)	Ω(log n/log log n)	
2-level rotated array	O(1)	$O(\sqrt{n})$	$O(\sqrt{n})$	O(n)
O(1)-level rotated array	O(1)	$O(n^{\varepsilon})$	$O(n^{\varepsilon})$	O(n)

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