Reading Material

KT Chapter 4 (Divide and Conquer), section 4.1 (A First Recurrence: The Mergesort Algorithm), 4.2 (Further Recurrence Relations), and 4.5 (Integer Multiplication).

If you have another edition of the book than the international one from 2014 Divide and Conquer might be in a different chapter (e.g. 5) , but the subsections are the same. In that case you should of course also solve the corresponding exercises (the numbers within the chapters are the same, e.g. if Divide and Conquer is chapter 5 in our book, you should solve exercise 5.1 instead of 4.1).

Exercises

1 Reccurrences Use both the *recursion tree method* and the *(partial) substitution method* to solve each of the following recurrences.

1.1 $T(n) \leq \begin{cases} 2T(n/4) + c\sqrt{n} & \text{for } n > 4 \\ c & \text{otherwise} \end{cases}$

1.2 $T(n) \leq \begin{cases} 2T(n/4) + cn & \text{for } n > 4 \\ c & \text{otherwise} \end{cases}$

1.3 $T(n) \leq \begin{cases} 2T(n/4) + cn^2 & \text{for } n > 4 \\ c & \text{otherwise} \end{cases}$

1.4 $T(n) \leq \begin{cases} T(\frac{3n}{4}) + cn & \text{for } n > 4 \\ c & \text{otherwise} \end{cases}$

1.5 $T(n) \leq \begin{cases} T(n/2) + T(n/3) + T(n/6) + cn & \text{for } n > 6 \\ c & \text{otherwise} \end{cases}$

2 Median Solve KT exercise 4.1 (5.1 in the American edition).

3 Chocolate Agency You are a consultant for a chocolate agency that buys (large) boxes of chocolate from producers and sells them to chocolate stores. The chocolate stores then uses the chocolate to make delicate filled chocolate they can sell. Consider a time period period of n days:

- On each day i there is exactly one producer that produces and sells the chocolate at a price of s_i.
- On each day j there is exactly one store willing to pay b_j to buy a fresh box of chocolate.
- The quality of the chocolate decreases each day after its prodcution day. Therefore, a store will only pay $b_j - 100\$$ for a box that is one day old, $b_j - 200\$$ for a box that is two days old, etc.

When matching a producer and a store, the agency earns the difference between the chocolate producer's selling price and the price the store buys it for. That is, the profit of the agency is $b_j - s_i - (j - i) \cdot 100\$$ for chocolate bought from a producer on day i and sold to a store on day j.

Your job is to devise an algorithm that decides on a pair of days i and j such that the agency maximizes their profit (obviously i must be at most j).

The input to the algorithm is n, the producers' sale prices $[s_1, s_2, \ldots, s_n]$, and the stores' buy prices $[b_1, b_2, \ldots, b_n]$.

3.1 Let $n = 6$, and consider the following example:

days	1	2	3	4	5	6
s_i	500$	150$	500$	300$	450$	200$
b_i	700$	300$	400$	300$	600$	300$

What is the maximum possible profit? Which pair of days gives the maixmum profit?

3.2 Given a day x describe an algorithm that in $O(n)$ time finds the best pair (i, j) where $i \leq x$ and $j > x$. The best pair is the pair that maximizes the profit. Remember to argue that your algorithm is correct.

3.3 Describe a divide-and-conquer algorithm that finds the pair of producers and stores that maximizes the profit. Remember to argue that your algorithm is correct.

3.4 Let $T(n)$ be the worst case running time of your algorithm. Give a recurrence for $T(n)$ (and explain why it is correct). What is the asymptotic running time of your algorithm (explain how you obtained the result)?

3.5 [†] Implement your algorithm on CodeJudge (see CodeJudge for input/ouput format).

4 Divide-and-conquer on trees Solve KT exercise 4.6 (5.6 in the American edition).

5 Divide-and-conquer on grid graphs Solve KT exercise 4.7 (5.7 in the American edition).

Puzzle of the week: The Switch The hangman summons his 100 prisoners, announcing that they may meet to plan a strategy, but will then be put in isolated cells, with no communication. He explains that he has set up a switch room which contains a single switch, which is either on or off. It is not known to the prisoners whether the switch initially is on or off. Also, the switch is not connected to anything, but a prisoner entering the room may see whether the switch is on or off (because the switch is up or down). Every once in a while, the hangman will let one arbitrary prisoner into the switch room. The prisoner may throw the switch (on to off, or vice versa), or leave the switch unchanged. Nobody but the prisoners will ever enter the switch room. The hangman promises to let any prisoner enter the room from time to time, arbitrarily often. That is, eventually, each prisoner has been in the room at least once, twice, a thousand times, any number you want. At any time, any prisoner may declare "We have all visited the switch room at least once". If the claim is correct, all prisoners will be released. If the claim is wrong, the hangman will execute his job (on all the prisoners). What's the strategy?