Network Flows

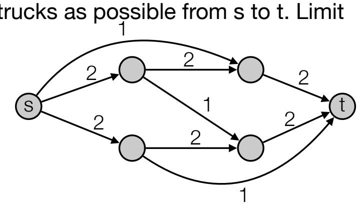
Inge Li Gørtz

Applications

- Matchings
- Job scheduling
- Image segmentation
- Baseball elimination
- Disjoint paths
- Survivable network design

Network Flow

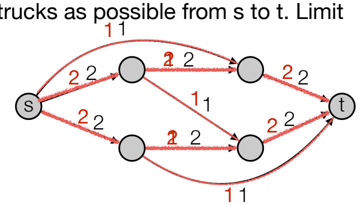
- Truck company: Wants to send as many trucks as possible from s to t. Limit of number of trucks on each road.



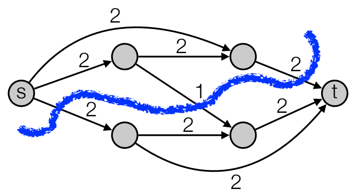
Network Flow

- Truck company: Wants to send as many trucks as possible from s to t. Limit of number of trucks on each road.

- Example 1:
    - Solution 1: 4 trucks
    - Solution 2: 5 trucks

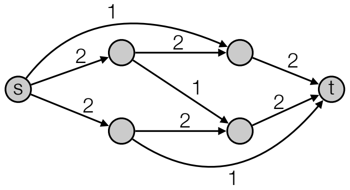


- Example 2:
    - 5 trucks (need to cross river).

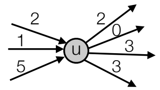


Network Flow

- Network flow:
  - graph G=(V,E).
  - Special vertices s (source) and t (sink).
  - s has no edges in and t has no edges out.
  - Every edge (e) has a (integer) capacity c(e) ≥ 0.
  - Flow:
    - **capacity constraint:** every edge e has a flow 0 ≤ f(e) ≤ c(e).
    - **flow conservation:** for all u ≠ s, t: flow into u equals flow out of u.



$$\sum_{v:(v,u)\in E} f(v,u) = \sum_{v:(u,v)\in E} f(u,v)$$



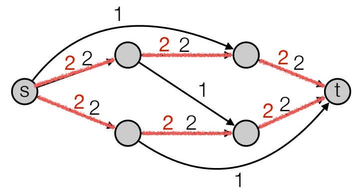
    - Value of flow f is the sum of flows out of s:

$$v(f) = \sum_{v:(s,v)\in E} f(e) = f^{out}(s)$$

  - **Maximum flow problem:** find s-t flow of maximum value

Algorithm

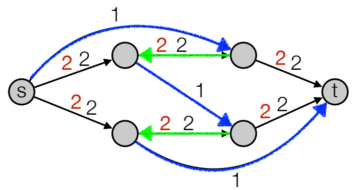
- Find path where we can send more flow.

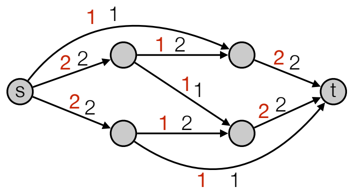


Algorithm

- Find path where we can send more flow.

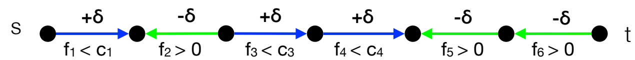
- Send flow back (cancel flow).



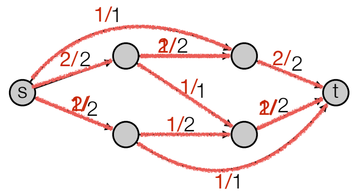


Augmenting Paths

- Augmenting path: s-t path P where
  - forward edges have leftover capacity
  - backwards edges have positive flow

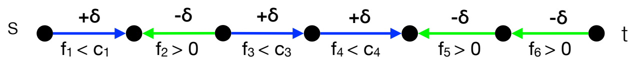


  - Can add extra flow: $\min(c_1 - f_1, f_2, c_3 - f_3, c_4 - f_4, f_5, f_6) = \delta = \text{bottleneck}(P)$.



Augmenting Paths

- Augmenting path (definition different than in CLRS): s-t path where
  - forward edges have leftover capacity
  - backwards edges have positive flow

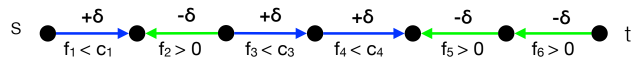


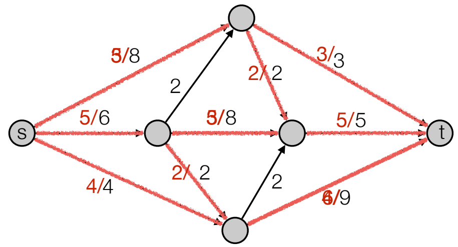
  - Can add extra flow: $\min(c_1 - f_1, f_2, c_3 - f_3, c_4 - f_4, f_5, f_6) = \delta$ = bottleneck(P).

- Ford-Fulkerson:
  - Find augmenting path, use it
  - Find augmenting path, use it
  - Find augmenting path, use it
  - ....

Ford Fulkerson

- Augmenting path: s-t path P where
  - forward edges have leftover capacity
  - backwards edges have positive flow

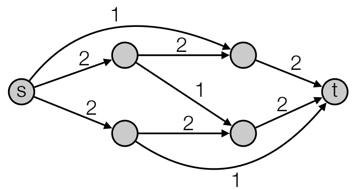


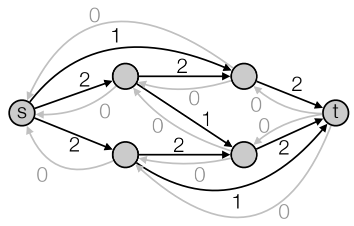


Analysis of Ford-Fulkerson

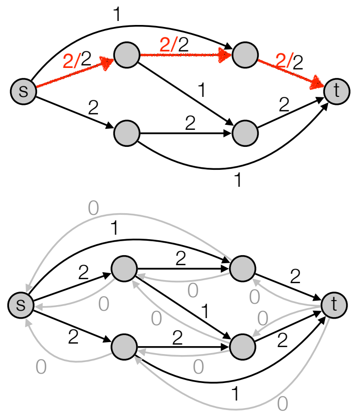
- Integral capacities implies theres is a maximum flow where all flow values f(e) are integers.
- Number of iterations:
  - Always increment flow by at least 1: #iterations ≤ max flow value f^*
- Time for one iteration:
  - Can find augmenting path in linear time: One iteration takes $O(m)$ time.
- Total running time = $O(|f^*|\, m)$.

Residual networks

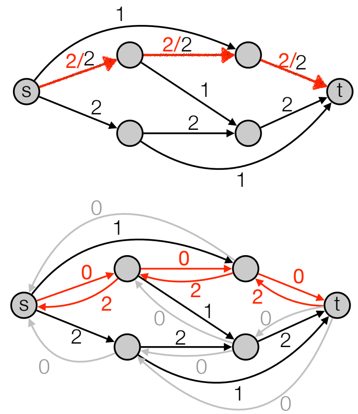




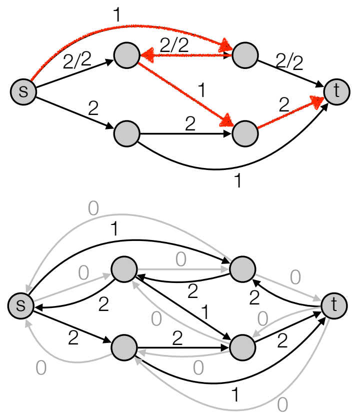
Residual networks



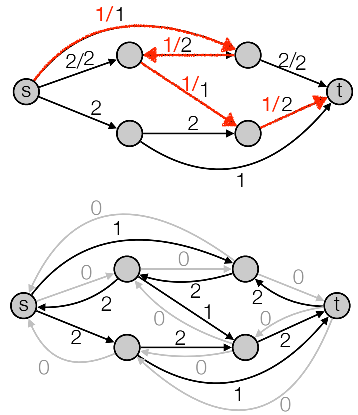
Residual networks



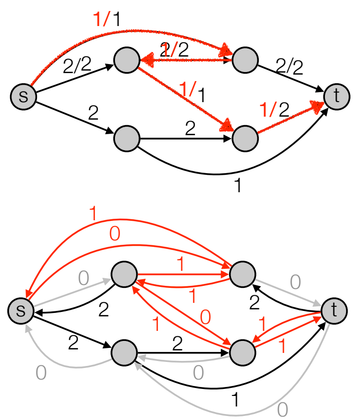
Residual networks



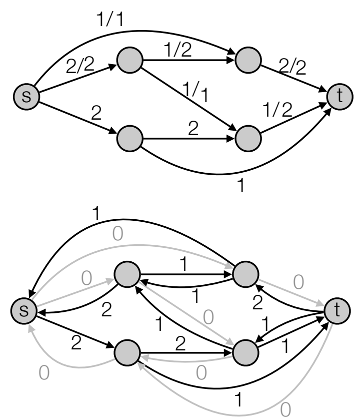
Residual networks



Residual networks

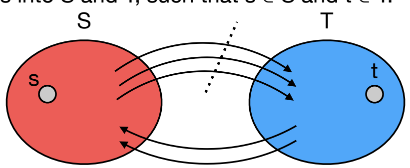


Residual networks

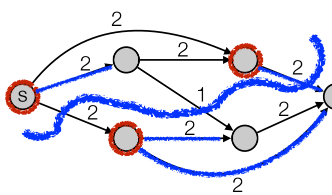


s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∊ S and t ∊ T.



- Capacity of cut: total capacity of edges going *from* S to T.

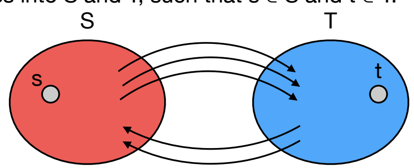


c(S,T) = 5

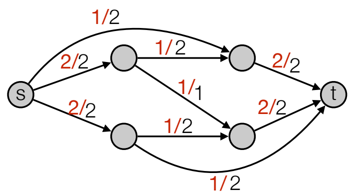
c(S,T) = 8

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∊ S and t ∊ T.

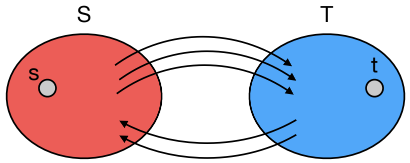


- Flow across cut: = flow *from* S to T *minus flow from* T to S.

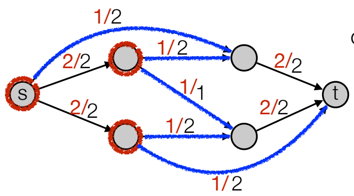


s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



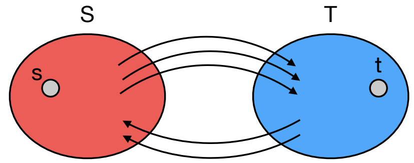
- Flow across cut: = flow *from* S to T *minus flow from* T to S.



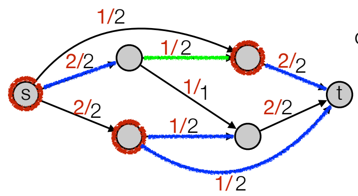
c(S,T) = 9 f(S,T) = 5

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



- Flow across cut: = flow *from* S to T *minus flow from* T to S.

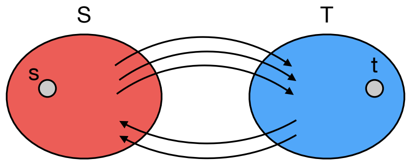


c(S,T) = 8

f(S,T) = 6 - 1 = 5

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



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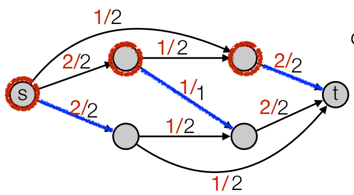
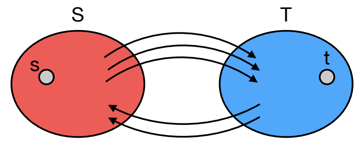


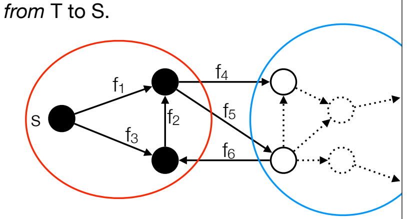
c(S,T) = 5 f(S,T) = 5

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



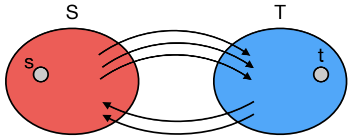
- Flow across cut = flow *from* S to T *minus flow from* T to S.
- Flow across cut: $f_4 + f_5 - f_6 = ?$
  - $f_4 + f_5 - f_1 - f_2 = 0$
  - $f_2 - f_6 - f_3 = 0$
  - $f_1 + f_3 = |f|$
  - $(f_4 + f_5 - \cancel{f_1} - \cancel{f_2}) + (\cancel{f_2} - f_6 - \cancel{f_3}) + (\cancel{f_1} + \cancel{f_3}) = |f|$
  - $f_4 + f_5 - f_6 = |f|$



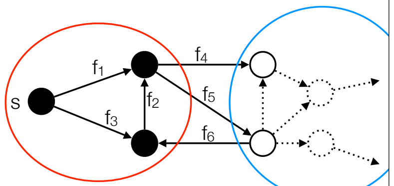
- Flow across cut is |f| for all cuts => flow out of s = flow into t.

s-t Cuts

- Cut: Partition of vertices into S and T, such that $s \in S$ and $t \in T$.

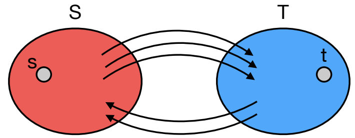


- Flow across cut is $|f|$ for all cuts => flow out of s = flow into t.
- $|f| \leq c(S,T)$:
  - $|f| = f_4 + f_5 - f_6 \leq f_4 + f_5 \leq c_4 + c_5 = c(S,T)$



s-t Cuts

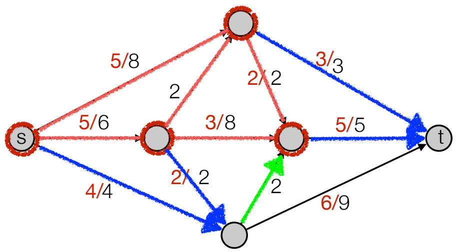
- Cut: Partition of vertices into S and T, such that $s \in S$ and $t \in T$.



- Suppose we have found flow f and cut (S,T) such that $|f| = c(S,T)$. Then f is a maximum flow and (S,T) is a minimum cut.
  - Let f* be the maximum flow and the (S*,T*) minimum cut:
  - $|f| \leq |f^*| \leq c(S^*,T^*) \leq c(S,T)$.
  - Since $|f| = c(S,T)$ this implies $|f| = |f^*|$ and $c(S,T) = c(S^*,T^*)$.

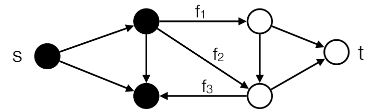
Finding minimum cuts

- Use Ford-Fulkerson to find a max-flow (finding augmenting paths).
- When no augmenting s-t path:
  - Let S be all vertices to which there exists an augmenting path from s.
  - All forward edges in the minimum cut are "full" (flow = capacity).
  - All backwards edges in minimum cut have 0 flow.
  - => value of flow (S,T) = capacity of the cut.



Use of Max-flow min-cut theorem

- There is no augmenting path <=> f is a maximum flow.
  - *f maximum flow => no augmenting path:*
    - Show that exists augmenting path => f not maximum flow.
  - *no augmenting path => f maximum flow*
    - no augmenting path => exists cut (S,T) where $|f| = c(S,T)$:
      - Let S be all vertices to which there exists an augmenting path from s.
      - t not in S (since there is no augmenting s-t path).
      - Edges from S to T: $f_1 = c_1$ and $f_2 = c_2$.
      - Edges from T to S: $f_3 = 0$.
      - => $|f| = f_1 + f_2 - f_3 = f_1 + f_2 = c_1 + c_2 = c(S,T)$.
      - => f a maximum flow and (S,T) a minimum cut.



Removing assumptions

- Edges into s and out of t:

$$v(f) = f^{out}(s) - f^{in}(s)$$

- Capacities not integers.

Network Flow

- Multiple sources and sinks: