Network Flows

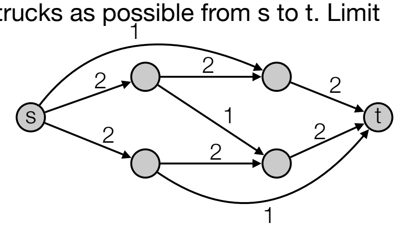
Inge Li Gørtz

Applications

- Matchings
- Job scheduling
- Image segmentation
- Baseball elimination
- Disjoint paths
- Survivable network design

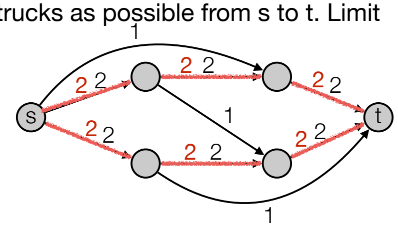
Network Flow

- Truck company: Wants to send as many trucks as possible from s to t. Limit of number of trucks on each road.



Network Flow

- Truck company: Wants to send as many trucks as possible from s to t. Limit of number of trucks on each road.
- Example 1:
  - Solution 1: 4 trucks



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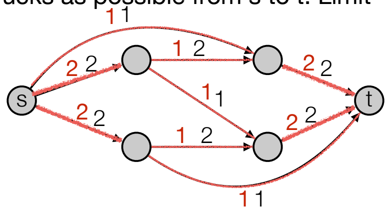

- Example 1:
  - Solution 1: 4 trucks
  - Solution 2: 5 trucks

Network Flow

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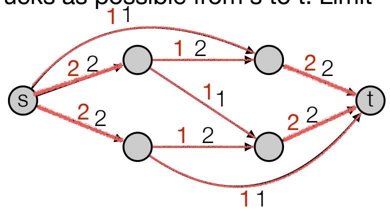
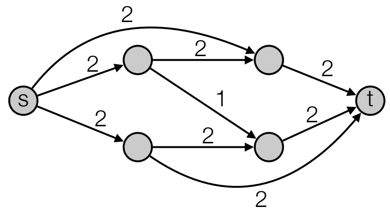


- Example 1:
  - Solution 1: 4 trucks
  - Solution 2: 5 trucks

- Example 2:



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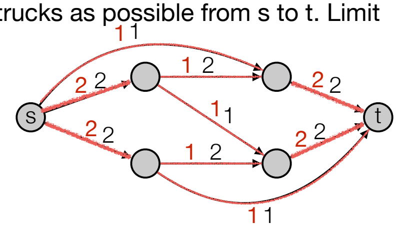
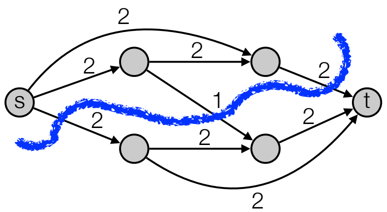


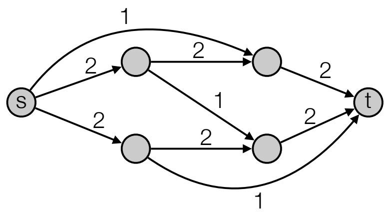
- Example 1:
  - Solution 1: 4 trucks
  - Solution 2: 5 trucks

- Example 2:
  - 5 trucks (need to cross river).

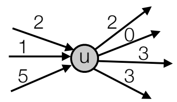


Network Flow

- Network flow:
  - graph G=(V,E).
  - Special vertices s (source) and t (sink).
  - s has no edges in and t has no edges out.
  - Every edge (e) has a (integer) capacity c(e) ≥ 0.
  - Flow:
    - **capacity constraint:** every edge e has a flow 0 ≤ f(e) ≤ c(e).
    - **flow conservation:** for all u ≠ s, t: flow into u equals flow out of u.



$$\sum_{v:(v,u)\in E} f(v,u) = \sum_{v:(u,v)\in E} f(u,v)$$



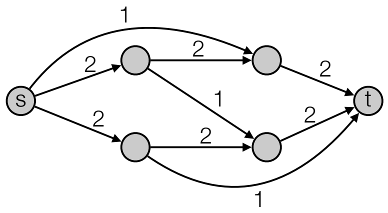
- Value of flow f is the sum of flows out of s:

$$v(f) = \sum_{v:(s,v)\in E} f(e) = f^{out}(s)$$

- **Maximum flow problem:** find s-t flow of maximum value

Algorithm

- Find path where we can send more flow.



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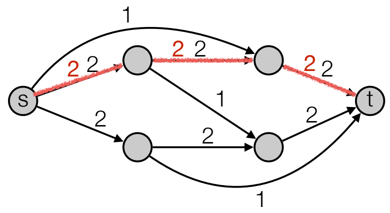

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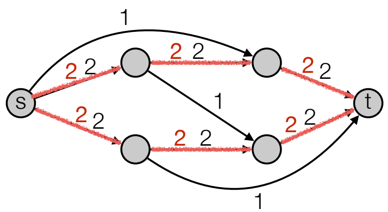

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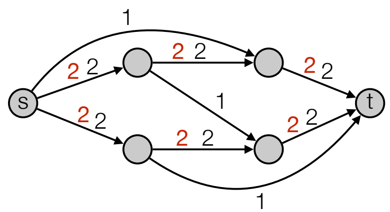
- Find path where we can send more flow.

- Send flow back (cancel flow).



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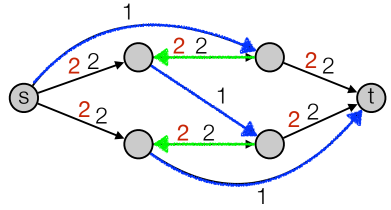

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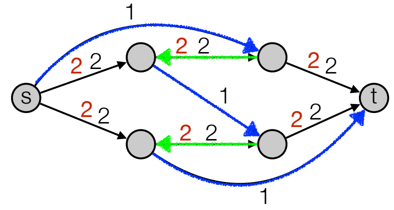
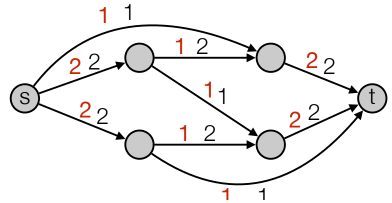
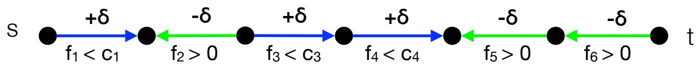


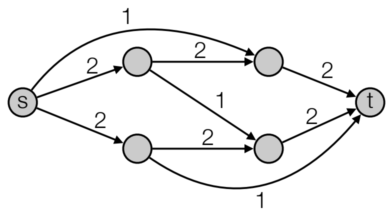


Augmenting Paths

- Augmenting path: s-t path P where
  - forward edges have leftover capacity
  - backwards edges have positive flow



  - Can add extra flow: $\min(c_1 - f_1, f_2, c_3 - f_3, c_4 - f_4, f_5, f_6) = \delta = \text{bottleneck}(P)$.



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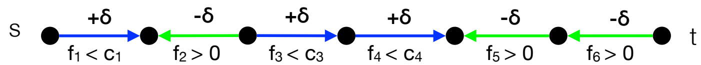

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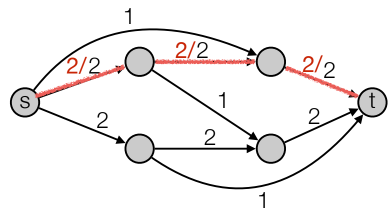

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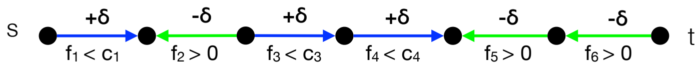

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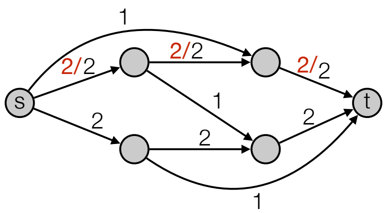

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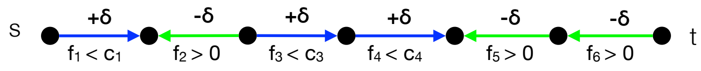

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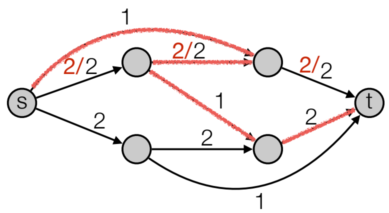

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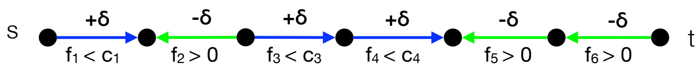

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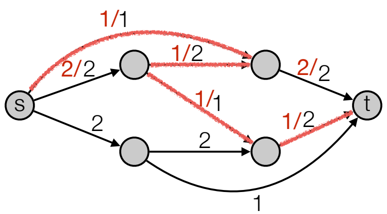

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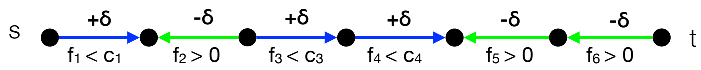

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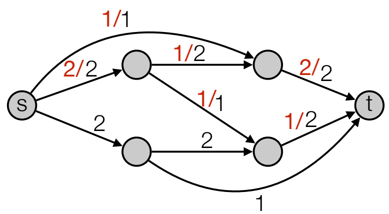

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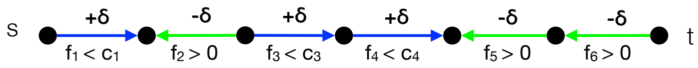

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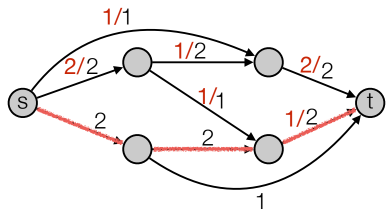

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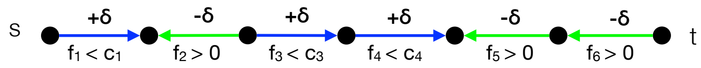

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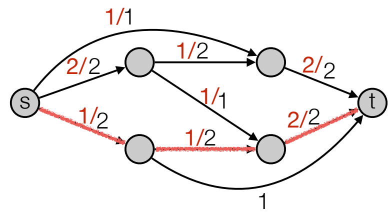

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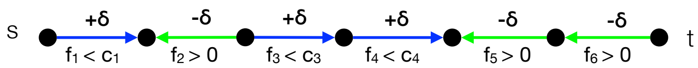

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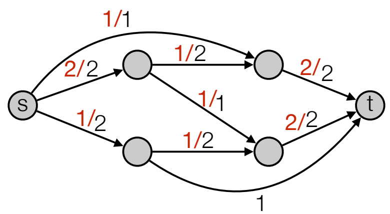

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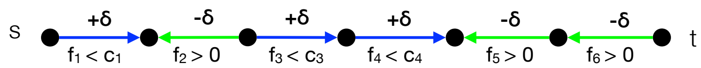

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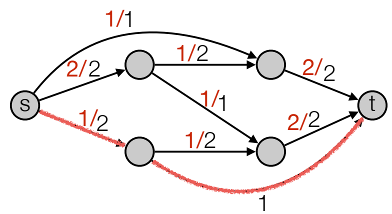

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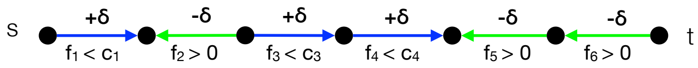

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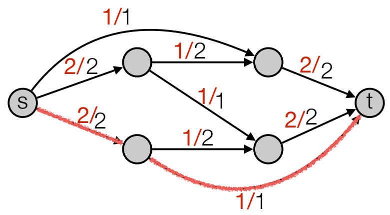

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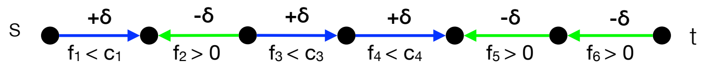

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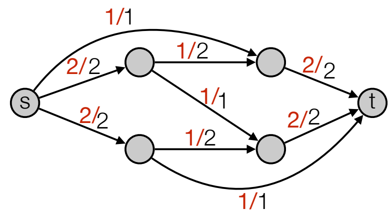

Augmenting Paths

- Augmenting path (definition different than in CLRS): s-t path where
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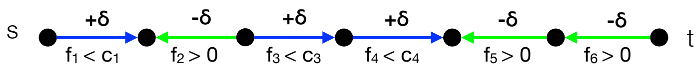

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- Ford-Fulkerson:
  - Find augmenting path, use it
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  - ....

Ford Fulkerson

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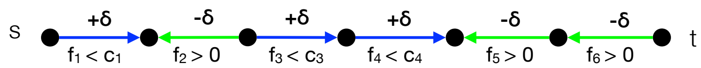

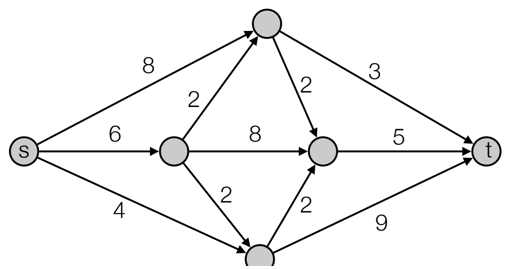

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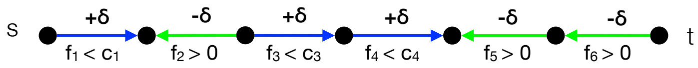

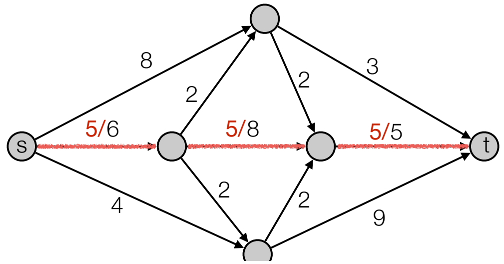

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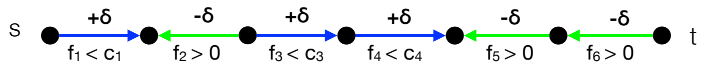

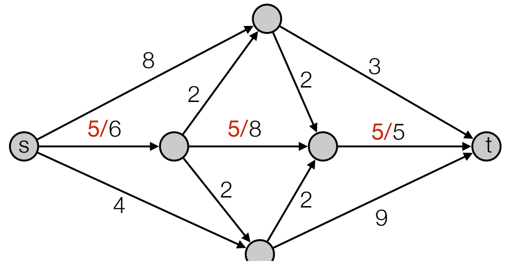

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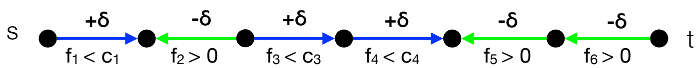

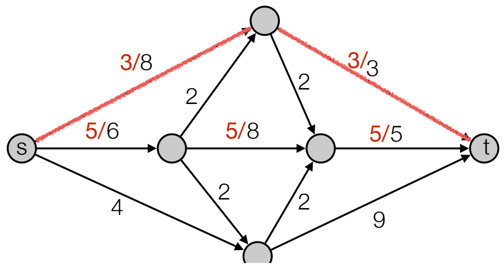

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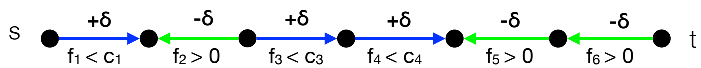

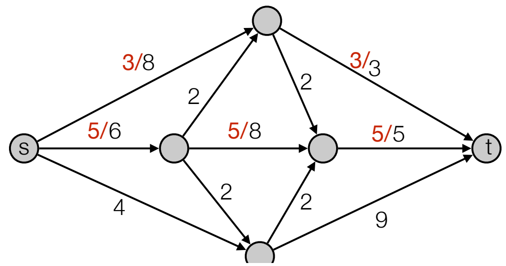

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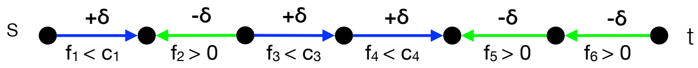

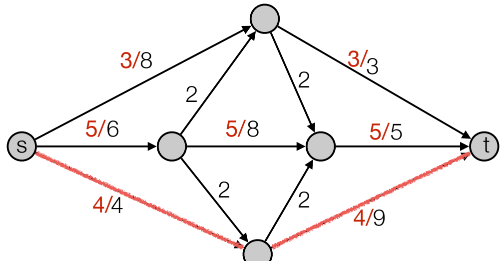

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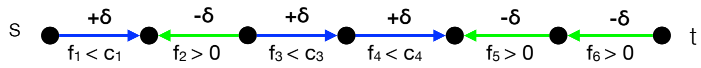

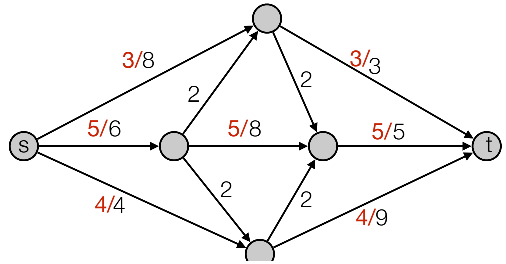

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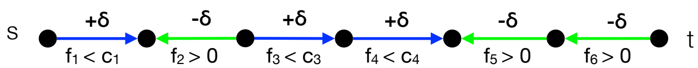
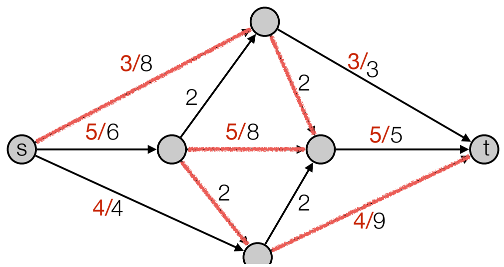
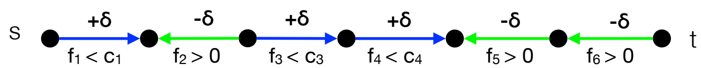
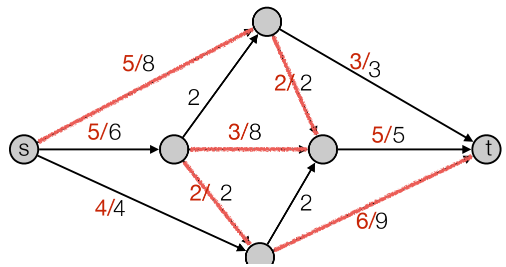




Ford Fulkerson

- Augmenting path: s-t path P where
  - forward edges have leftover capacity
  - backwards edges have positive flow





Ford Fulkerson

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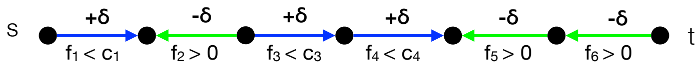
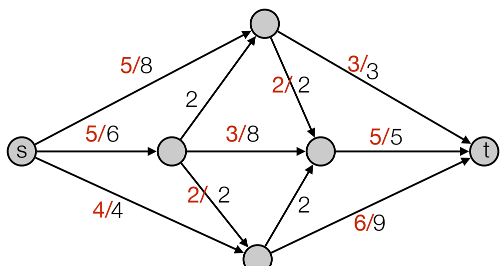
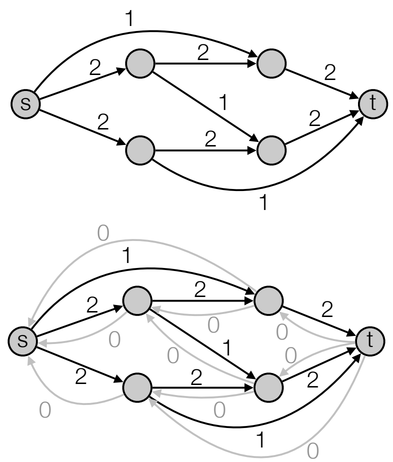




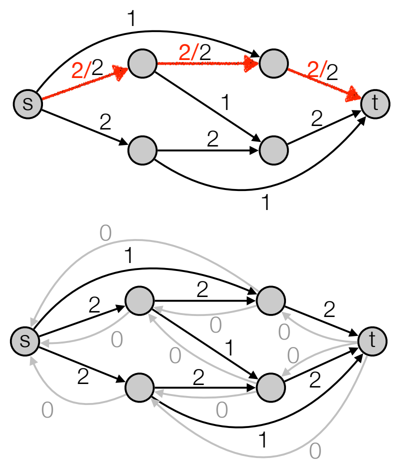
Analysis of Ford-Fulkerson

- Integral capacities implies theres is a maximum flow where all flow values f(e) are integers.
- Number of iterations:
  - Always increment flow by at least 1: #iterations ≤ max flow value f*
- Time for one iteration:
  - Can find augmenting path in linear time: One iteration takes O(m) time.
- Total running time = O(|f*| m).

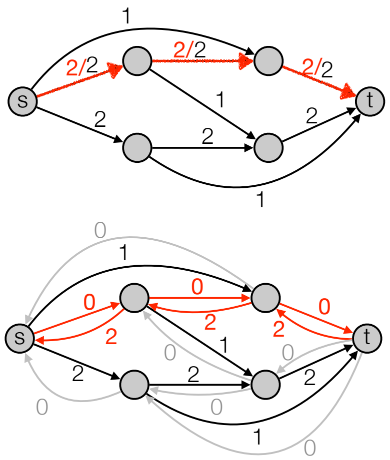
Residual networks



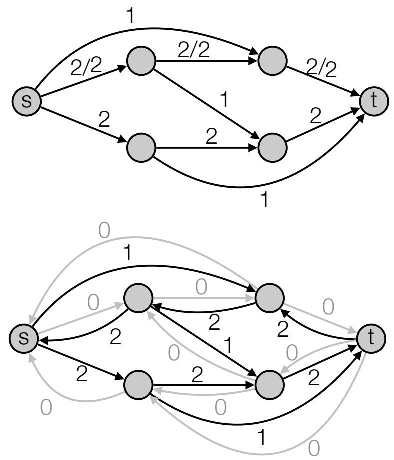
Residual networks



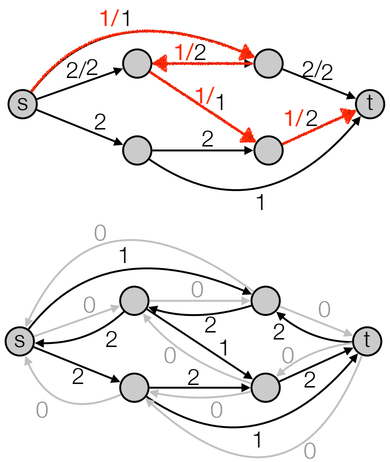
Residual networks



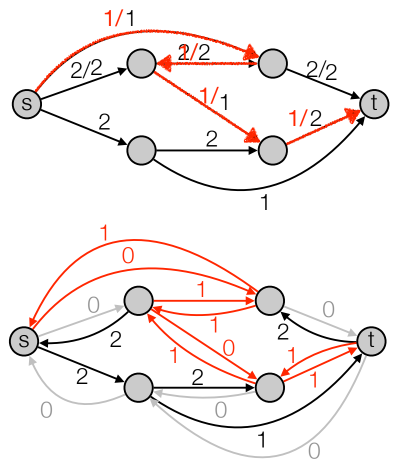
Residual networks



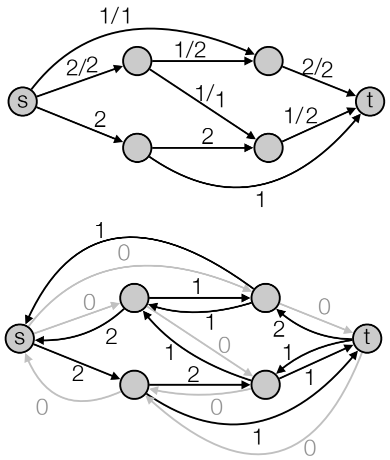
Residual networks



Residual networks



Residual networks

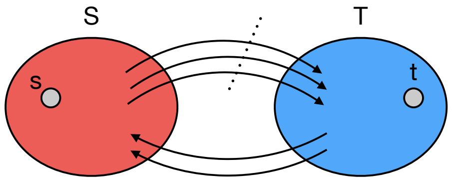


Implementation

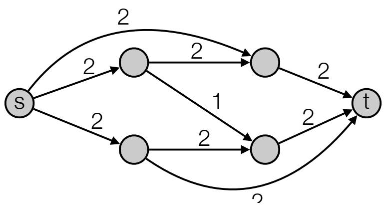
```
adj[0...n-1]                    # adjacency list
cap                             # capacity dictionary

for each edge (u,v,c):
    adj[u].append(v)            # add v to u's adjacency list (adding the edge u -> v)
    adj[v].append(u)            # add u to v's adjacency list (adding the edge v -> u)
    cap[(u,v)] = c              # set capacity on u->v edge to c.
    cap[(v,u)] = 0              # set capacity on u->v edge to 0.

# Graph search algorithm that searches for an augmenting path from u->v      (e.g. BFS or DFS)
AugPath():
    visited[0...n-1]            # visited list initialized to False
    pred[0...n-1]               # predecessor list
    stack S                     # initialize stack S

    push(S,s) and set visited[s] = True
    while S not empty and not visited[t]:
        u = pop(S)
        for v in adj[u]:
            if visited[v] or cap[(u,v)] = 0:
                continue
            visited[v] = True
            pred[v] = u
            push(S,v)
    if visited[t]:              # found augmenting path
        follow pred pointers back from t to s to find delta          (fill out details yourself)
        follow pred pointers back from t to s to update capacities   (fill out details yourself)
        return delta
    return 0                    # no augmenting path found
```

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



- Capacity of cut: total capacity of edges going *from* S to T.



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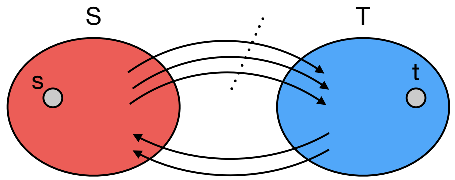
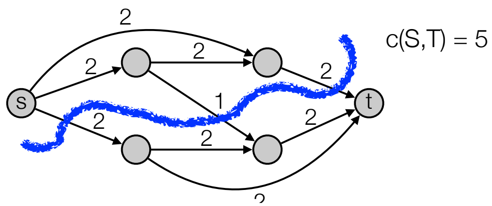


- Capacity of cut: total capacity of edges going *from* S to T.



s-t Cuts

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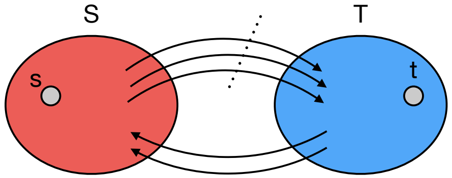
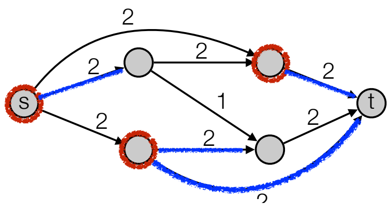


- Capacity of cut: total capacity of edges going *from* S to T.



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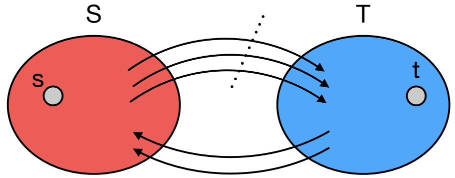
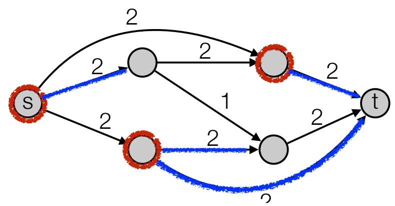


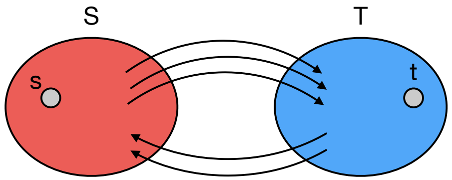
- Capacity of cut: total capacity of edges going *from* S to T.



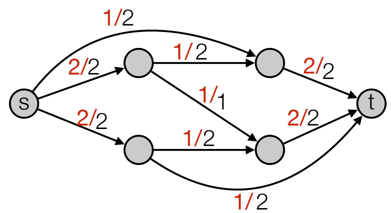
c(S,T) = 8

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



- Flow across cut: = flow *from* S to T *minus flow from* T to S.



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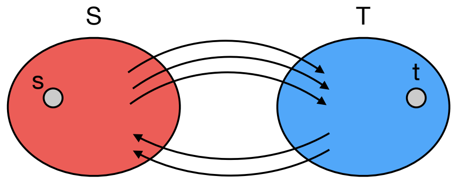

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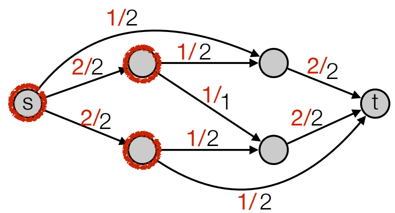

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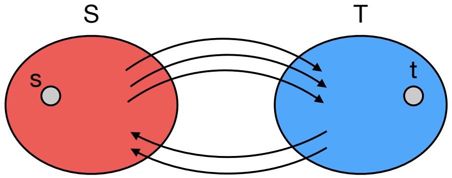
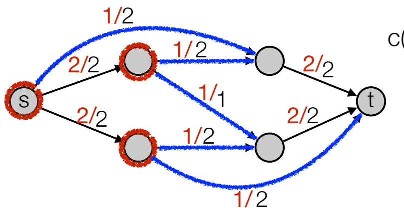


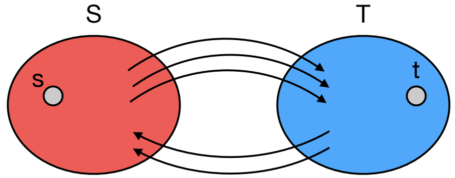
- Flow across cut: = flow *from* S to T *minus flow from* T to S.



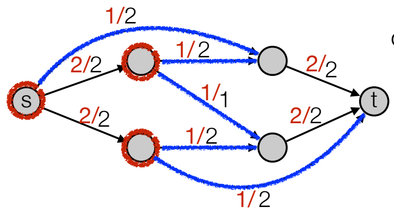
c(S,T) = 9

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



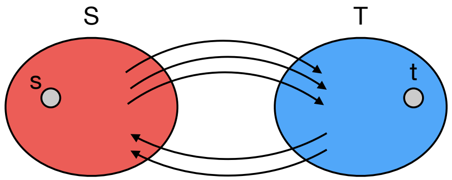
- Flow across cut: = flow *from* S to T *minus flow from* T to S.



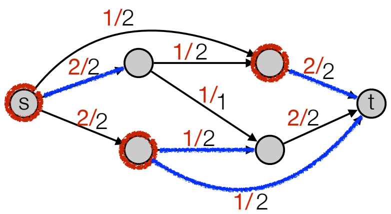
c(S,T) = 9   f(S,T) = 5

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



- Flow across cut: = flow *from* S to T *minus flow from* T to S.



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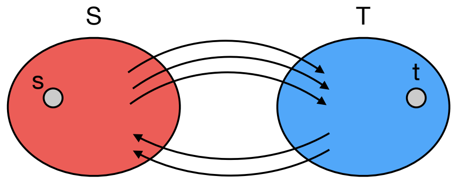
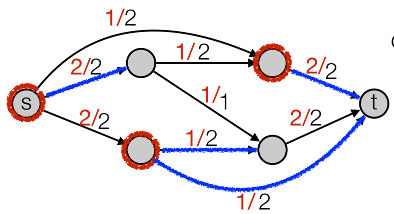


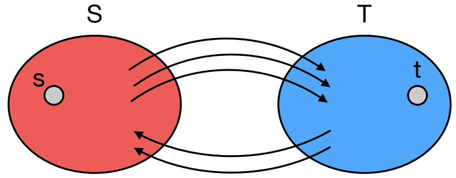
- Flow across cut: = flow *from* S to T *minus flow from* T to S.



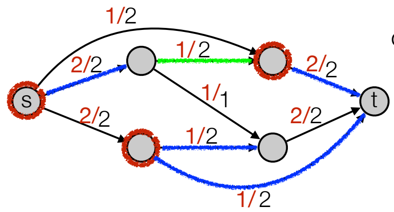
c(S,T) = 8

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



- Flow across cut: = flow *from* S to T *minus flow from* T to S.

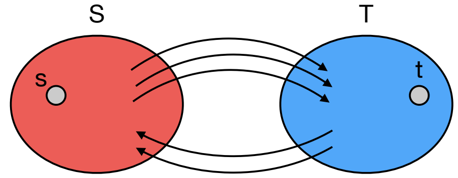


c(S,T) = 8

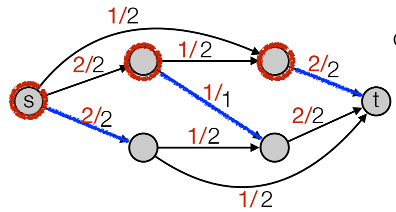
f(S,T) = 6 - 1 = 5

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



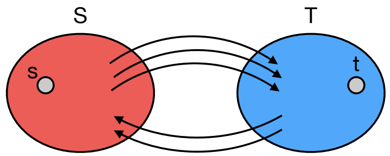
- Flow across cut: = flow *from* S to T *minus flow from* T to S.



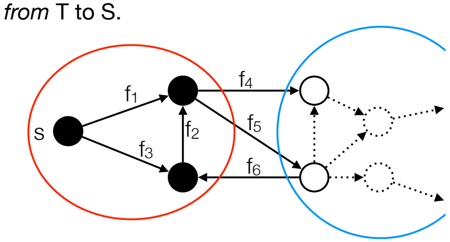
c(S,T) = 5 f(S,T) = 5

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



- Flow across cut = flow *from* S to T *minus flow from* T to S.



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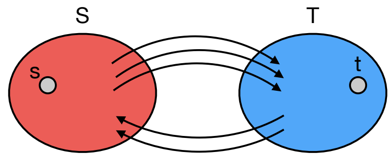
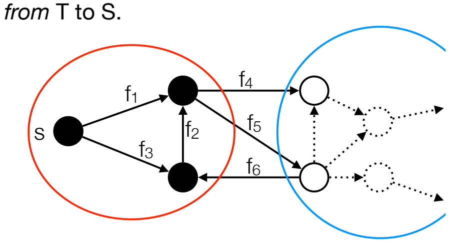


- Flow across cut = flow *from* S to T *minus flow from* T to S.
- Flow across cut: $f_4 + f_5 - f_6$ = ?



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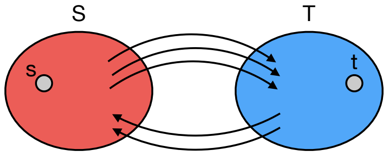
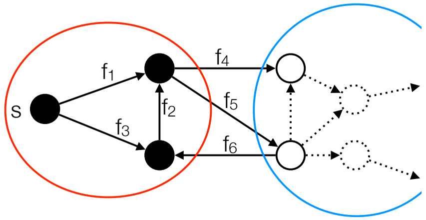


- Flow across cut = flow *from* S to T *minus flow from* T to S.
- Flow across cut: $f_4 + f_5 - f_6 = ?$
  - $f_4 + f_5 - f_1 - f_2 = 0$
  - $f_2 - f_6 - f_3 = 0$
  - $f_1 + f_3 = |f|$
  - $(f_4 + f_5 - f_1 - f_2) + (f_2 - f_6 - f_3) + (f_1 + f_3) = |f|$



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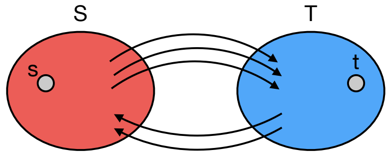

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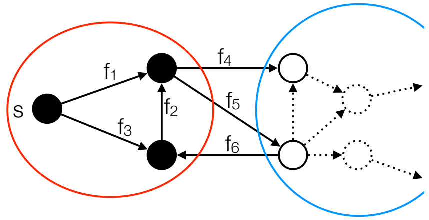
  - $f_2 - f_6 - f_3 = 0$
  - $f_1 + f_3 = |f|$
  - $(f_4 + f_5 - \cancel{f_1} - \cancel{f_2}) + (\cancel{f_2} - f_6 - \cancel{f_3}) + (\cancel{f_1} + \cancel{f_3}) = |f|$



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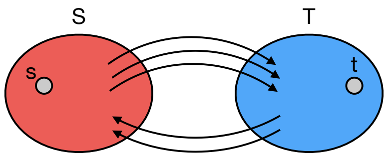

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  - $(f_4 + f_5 - \cancel{f_1} - \cancel{f_2}) + (\cancel{f_2} - f_6 - \cancel{f_3}) + (\cancel{f_1} + \cancel{f_3}) = |f|$
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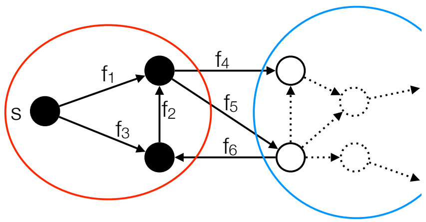

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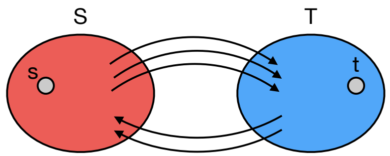

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  - $f_4 + f_5 - f_1 - f_2 = 0$
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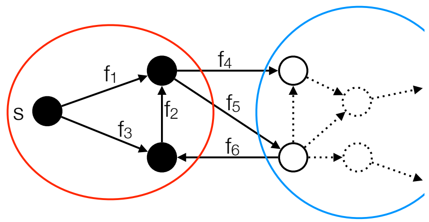
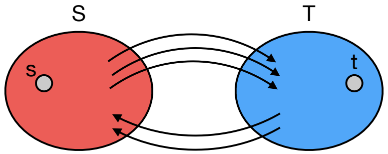
  - $f_1 + f_3 = |f|$
  - $(f_4 + f_5 - \cancel{f_1} - \cancel{f_2}) + (\cancel{f_2} - f_6 - \cancel{f_3}) + (\cancel{f_1} + \cancel{f_3}) = |f|$
  - $f_4 + f_5 - f_6 = |f|$



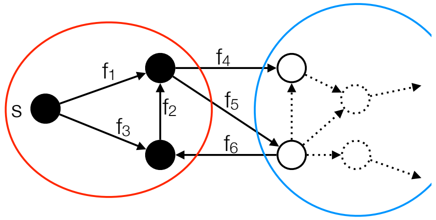
- Flow across cut is |f| for all cuts => flow out of s = flow into t.

s-t Cuts

- Cut: Partition of vertices into S and T, such that $s \in S$ and $t \in T$.



- Flow across cut is |f| for all cuts => flow out of s = flow into t.
- $|f| \leq c(S,T)$:
  - $|f| = f_4 + f_5 - f_6 \leq f_4 + f_5 \leq c_4 + c_5 = c(S,T)$



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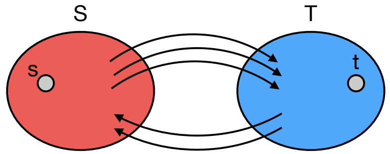
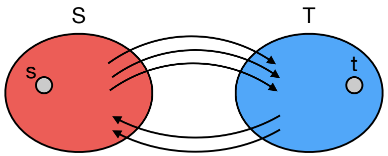
- Cut: Partition of vertices into S and T, such that $s \in S$ and $t \in T$.



- Suppose we have found flow f and cut (S,T) such that |f| = c(S,T). Then f is a maximum flow and (S,T) is a minimum cut.

s-t Cuts

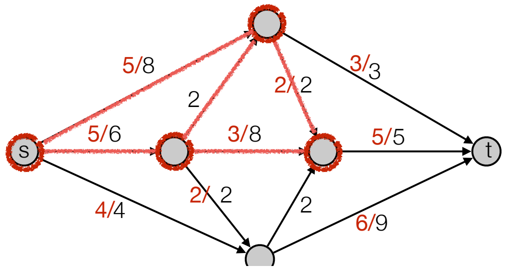
- Cut: Partition of vertices into S and T, such that $s \in S$ and $t \in T$.



- Suppose we have found flow f and cut (S,T) such that |f| = c(S,T). Then f is a maximum flow and (S,T) is a minimum cut.
  - Let f* be the maximum flow and the (S*,T*) minimum cut:
  - $|f| \leq |f^*| \leq c(S^*,T^*) \leq c(S,T)$.
  - Since |f| = c(S,T) this implies |f| = |f*| and c(S,T) = c(S*,T*).

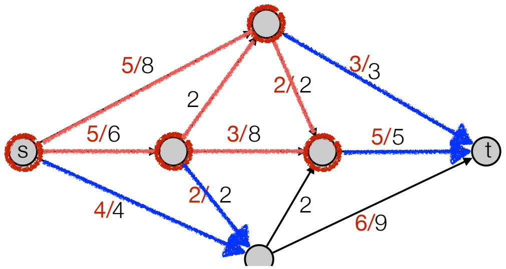
Finding minimum cuts

- Use Ford-Fulkerson to find a max-flow (finding augmenting paths).
- When no augmenting s-t path:
  - Let S be all vertices to which there exists an augmenting path from s.



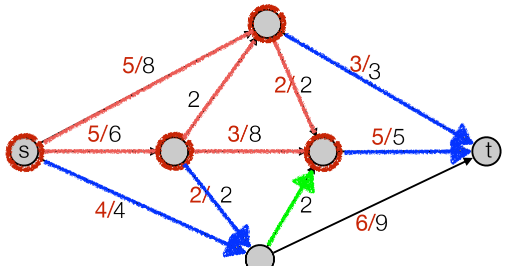
Finding minimum cuts

- Use Ford-Fulkerson to find a max-flow (finding augmenting paths).
- When no augmenting s-t path:
  - Let S be all vertices to which there exists an augmenting path from s.
  - value of flow (S,T) = capacity of the cut:
    - All forward edges in the minimum cut are "full" (flow = capacity).



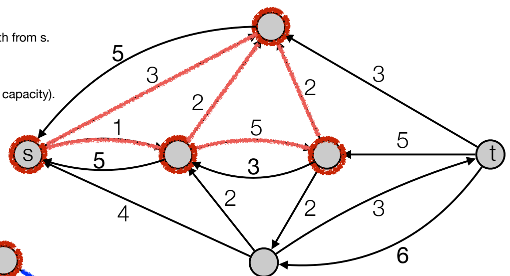
Finding minimum cuts

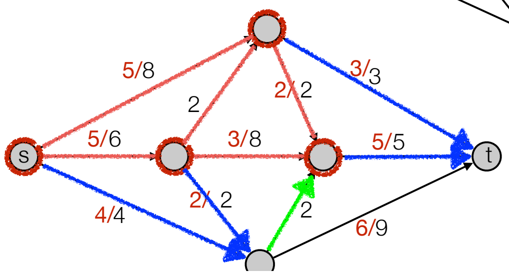
- Use Ford-Fulkerson to find a max-flow (finding augmenting paths).
- When no augmenting s-t path:
  - Let S be all vertices to which there exists an augmenting path from s.
  - value of flow (S,T) = capacity of the cut:
    - All forward edges in the minimum cut are "full" (flow = capacity).
    - All backwards edges in minimum cut have 0 flow.



Finding minimum cuts (with residual network).

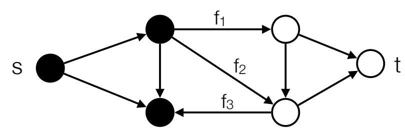
- Use Ford-Fulkerson to find a max-flow (finding augmenting paths).
- When no augmenting s-t path:
  - Let S be all vertices to which there exists an augmenting path from s.
  - value of flow (S,T) = capacity of the cut:
    - All forward edges in the minimum cut are "full" (flow = capacity).
    - All backwards edges in minimum cut have 0 flow.





Use of Max-flow min-cut theorem

- There is no augmenting path <=> f is a maximum flow.
  - *f maximum flow => no augmenting path:*
    - Show that exists augmenting path => f not maximum flow.
  - *no augmenting path => f maximum flow*
    - no augmenting path => exists cut (S,T) where |f| = c(S,T):
      - Let S be all vertices to which there exists an augmenting path from s.
      - t not in S (since there is no augmenting s-t path).
      - Edges from S to T: $f_1 = c_1$ and $f_2 = c_2$.
      - Edges from T to S: $f_3 = 0$.
      - => $|f| = f_1 + f_2 - f_3 = f_1 + f_2 = c_1 + c_2 = c(S,T)$.
      - => f a maximum flow and (S,T) a minimum cut.



Removing assumptions

- Edges into s and out of t:

$$v(f) = f^{out}(s) - f^{in}(s)$$

- Capacities not integers.

Network Flow

- Multiple sources and sinks: