Amortized Analysis and Splay Trees

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CLRS Chapter 17
Jeff Erickson notes

Today

- Amortized analysis
  - Multipop-stack
  - Dynamic tables
  - Splay trees

Dynamic tables

- Problem. Have to assign size of table at initialization.
- Goal. Only use space $\Theta(n)$ for an array with n elements.

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- Can insert and delete elements at the end.

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- Goal. Ensure size of array does not change to often.

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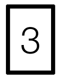

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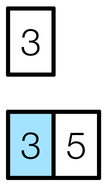

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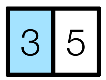

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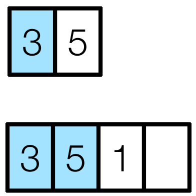

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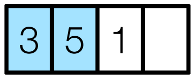

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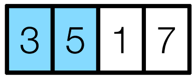

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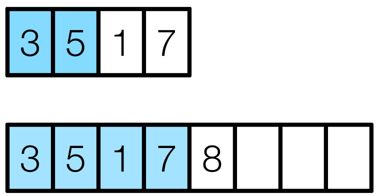

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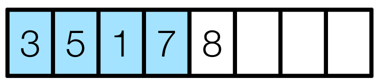

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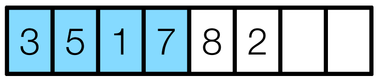

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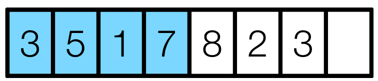

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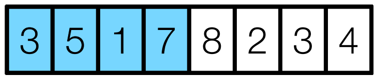

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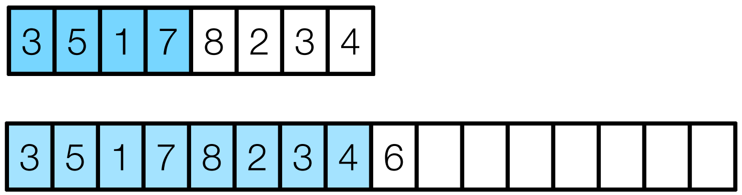

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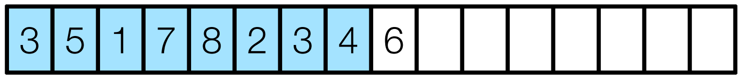

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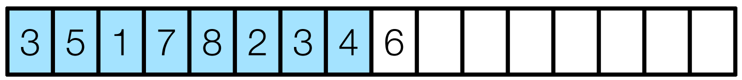

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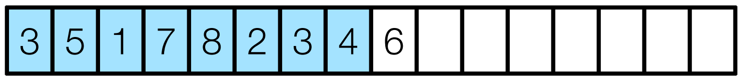

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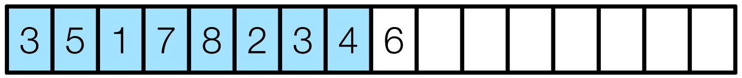

- Consequence. Insertion of n elements take time:
    - n + number of reinsertions = $n + 1 + 2 + 4 + 8 + \cdots + 2^{\log n} < 3n$.
    - Space: $\Theta(n)$.

Amortized Analysis

- Amortized analysis.
  - Average running time per operation over a *worst-case* sequence of operations.
- Methods.
  - Summation (aggregate) method
  - Accounting (tax) method
  - Potential method

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  - Amortized cost = total cost/#operations.

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- Analysis of doubling strategy (without deletions):
  - Total cost: $n + 1 + 2 + 4 + ... + 2^{\log n} = \Theta(n)$.
  - Amortized cost per insert: $\Theta(1)$.

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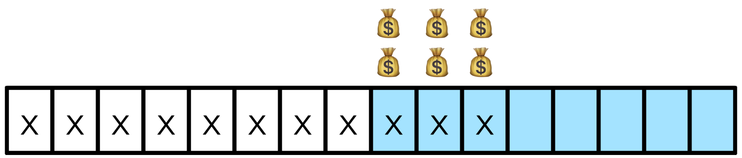

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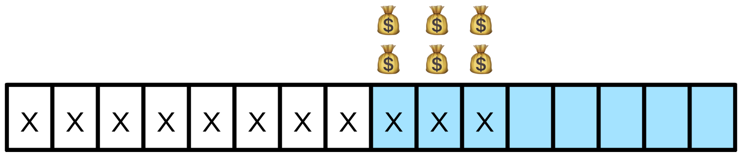

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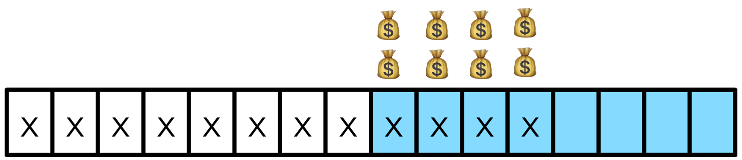

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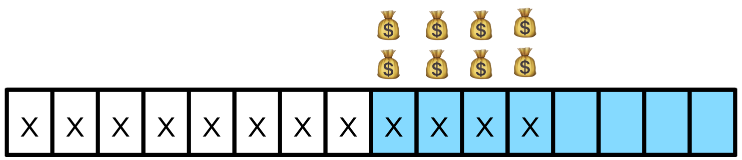

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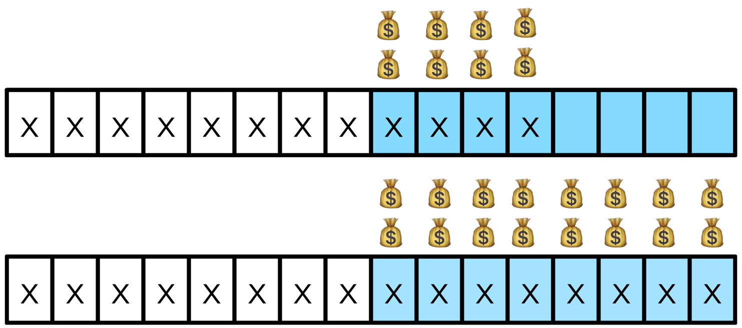

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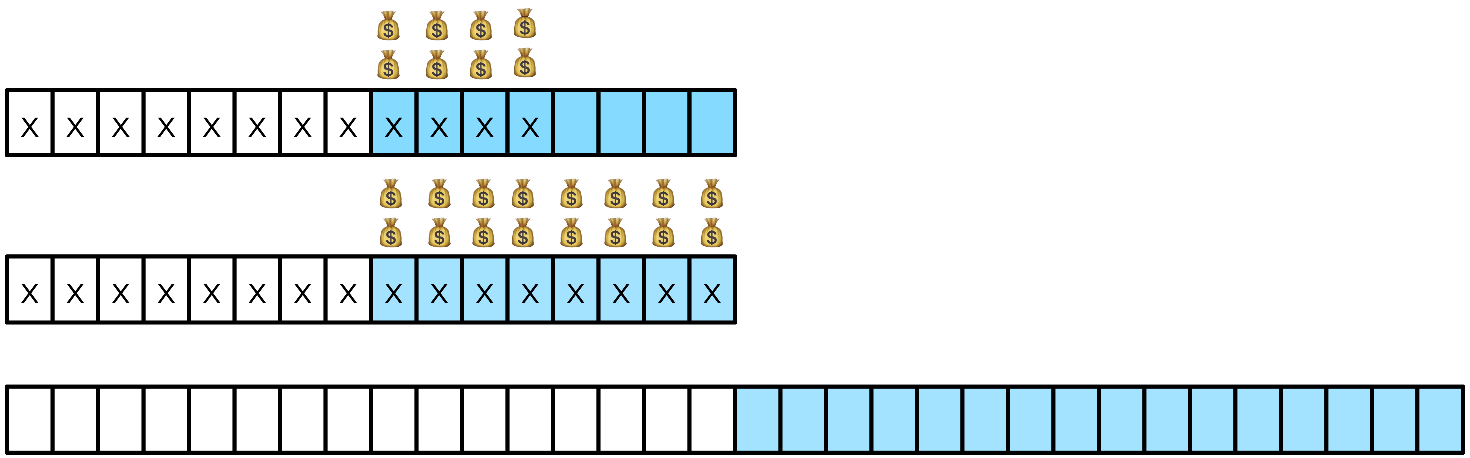

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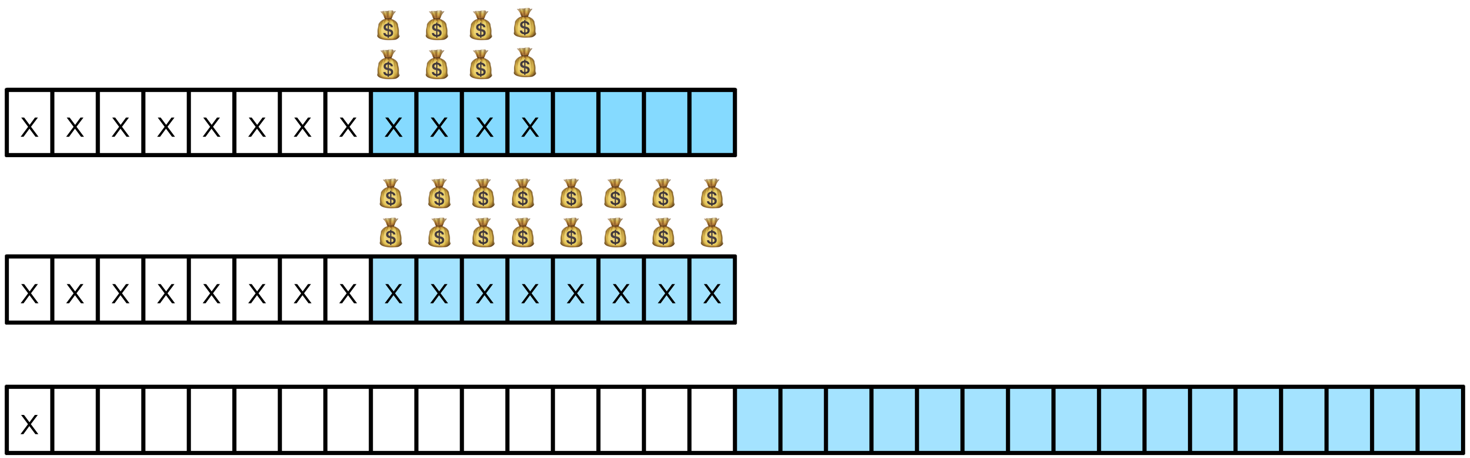

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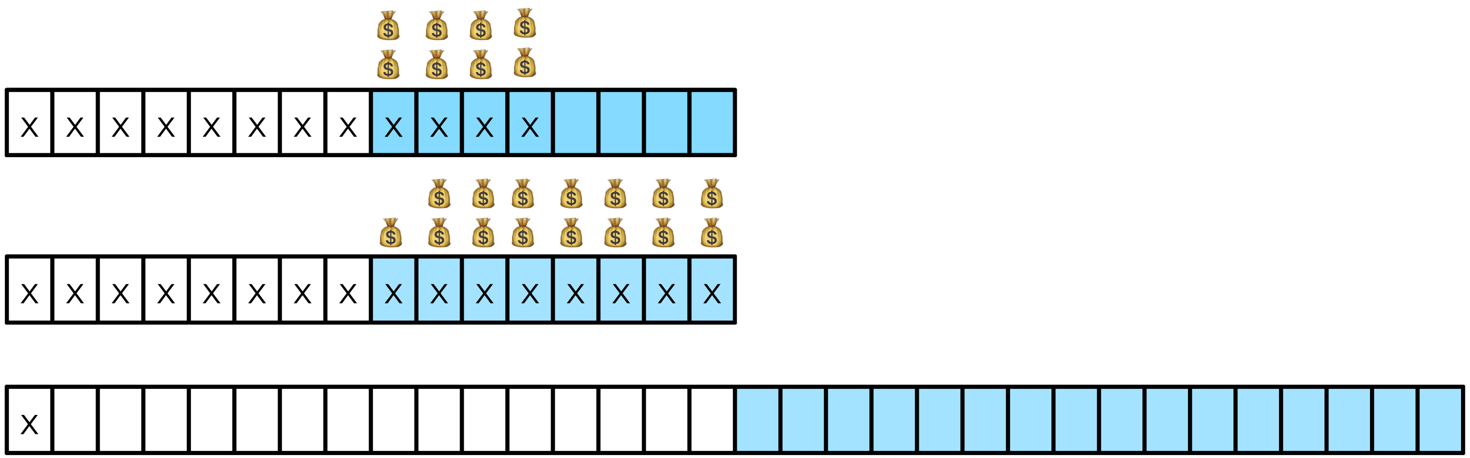

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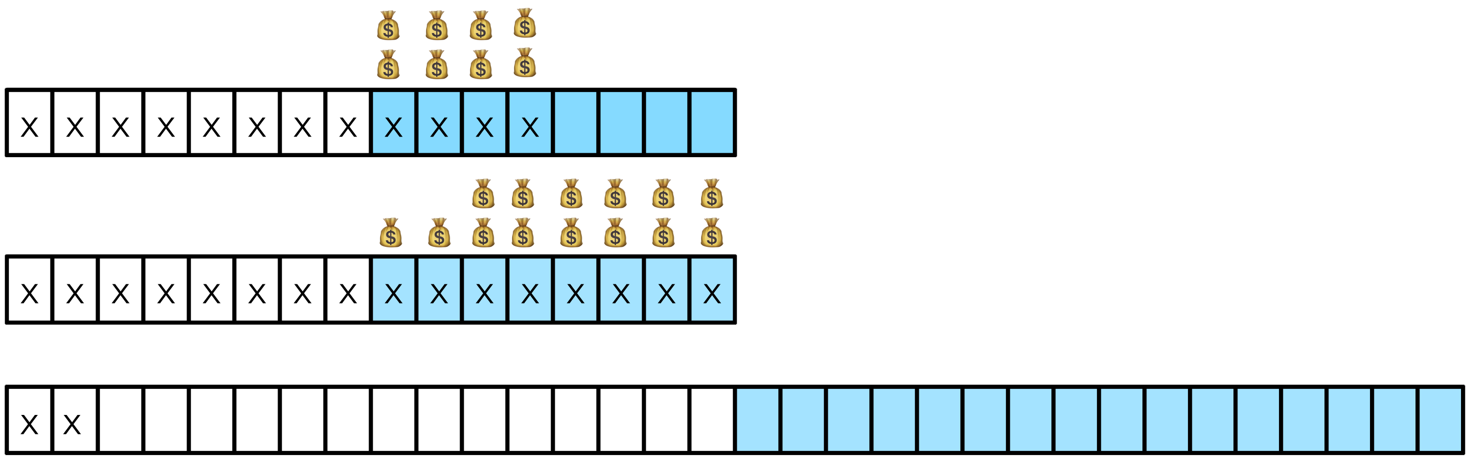

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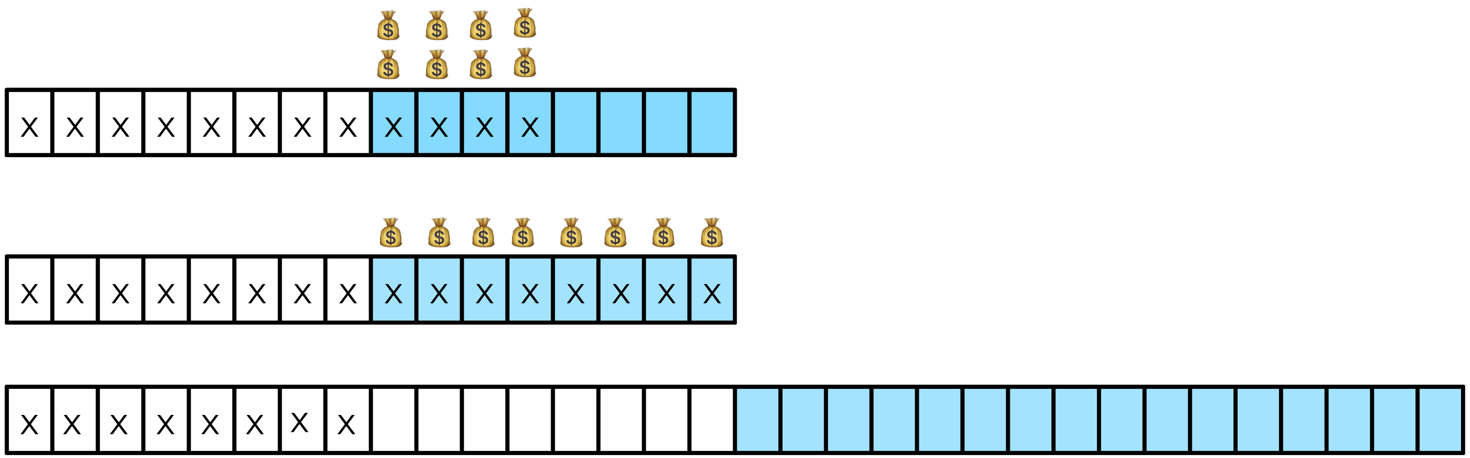

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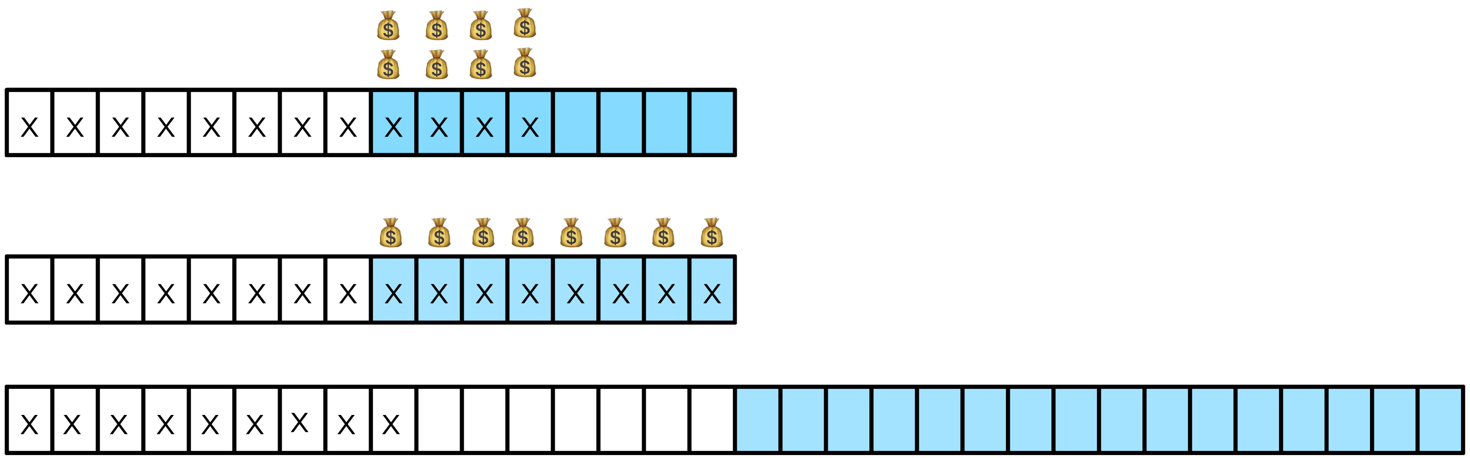

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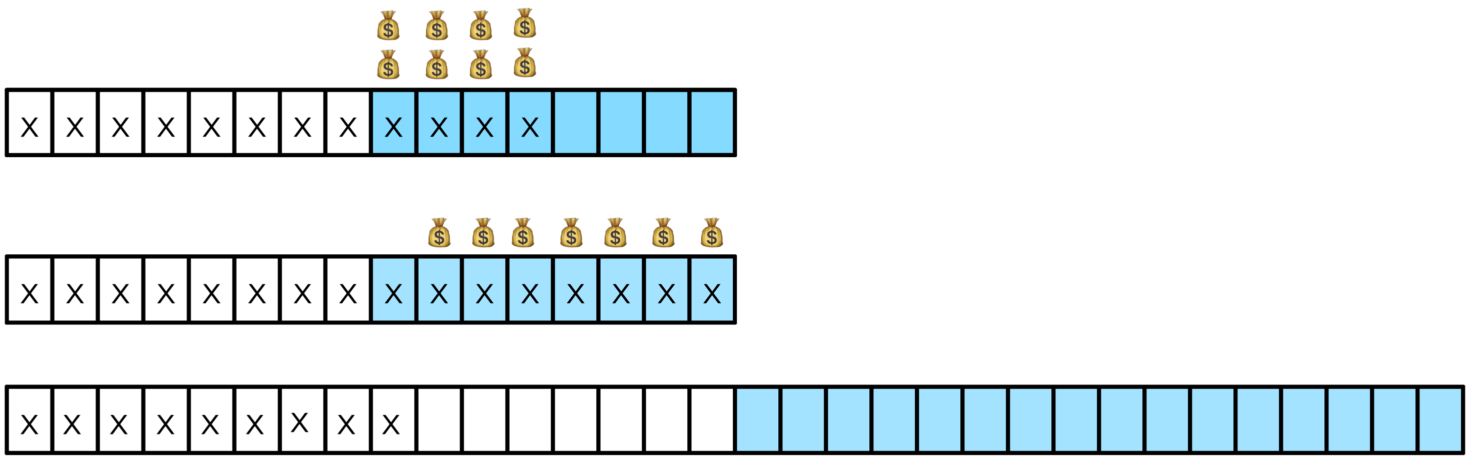

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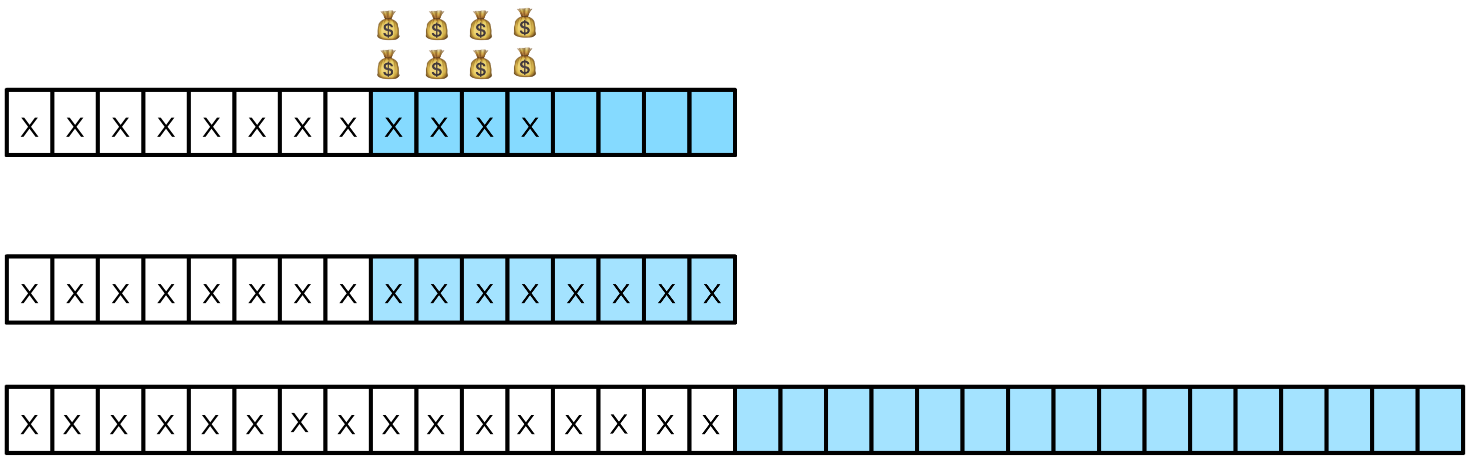

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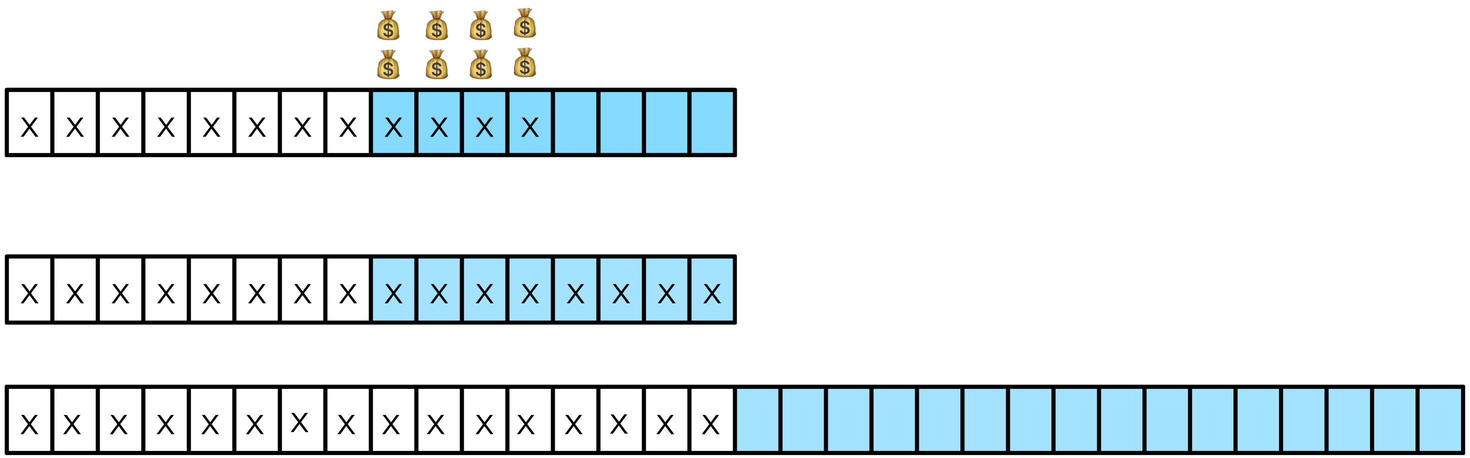
  - Amortized cost per operation: 3.



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  - => Total amortized cost an upper bound on the actual cost.

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  - Push(e): push element e onto stack.
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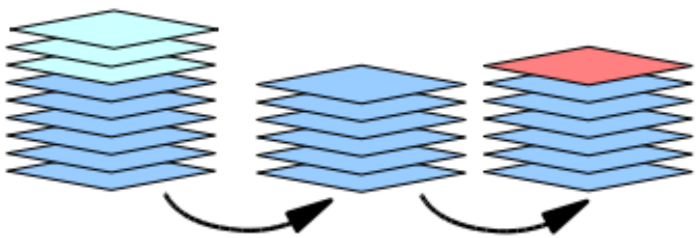

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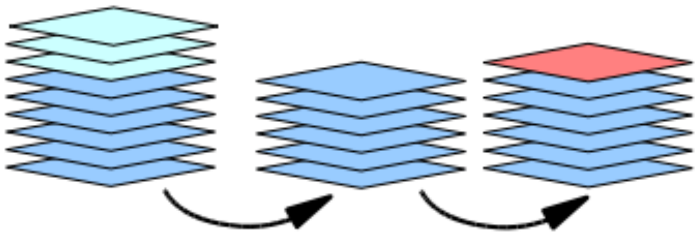

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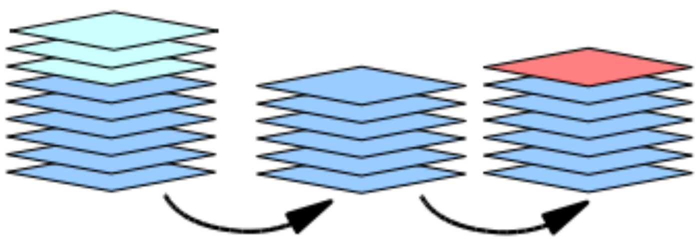

Stack: Aggregate Analysis

- Amortized analysis. Sequence of n Push and MultiPop operations.
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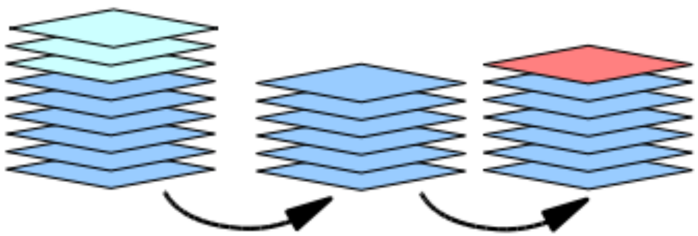

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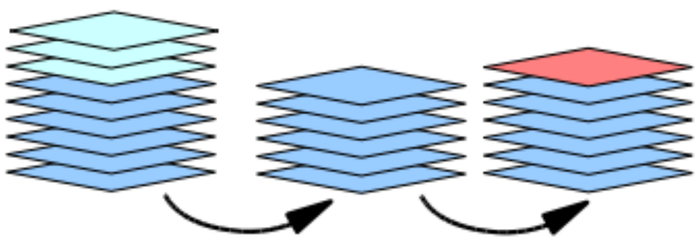

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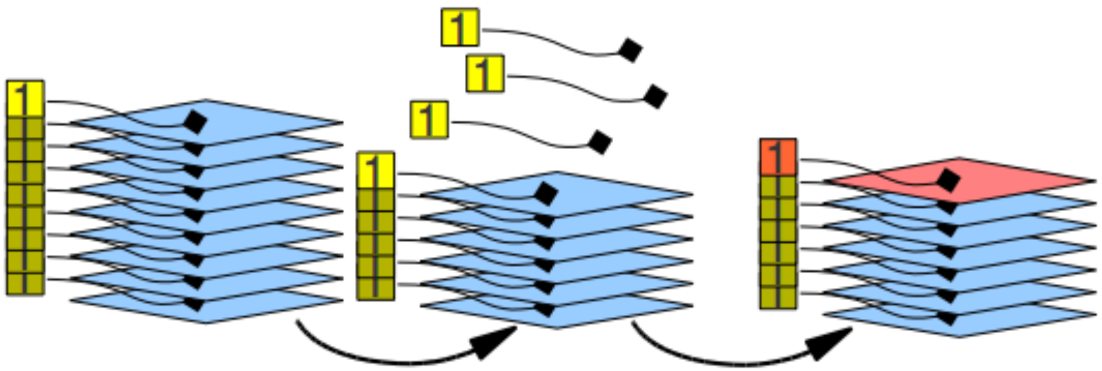

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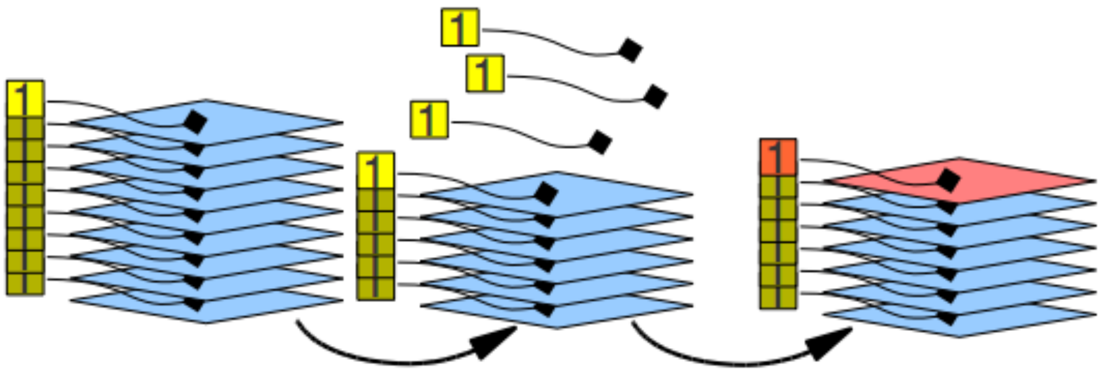
  - MultiPop: 1 (to pay for pop on empty stack).



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3	5	1	7	8	2	3	4	6							

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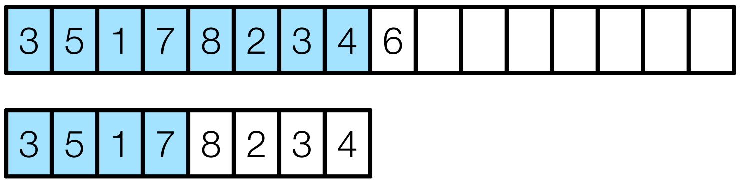

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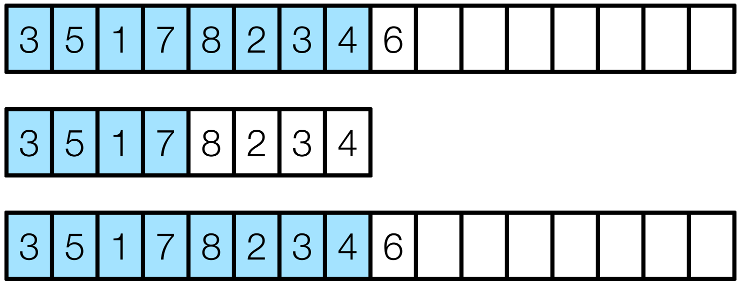

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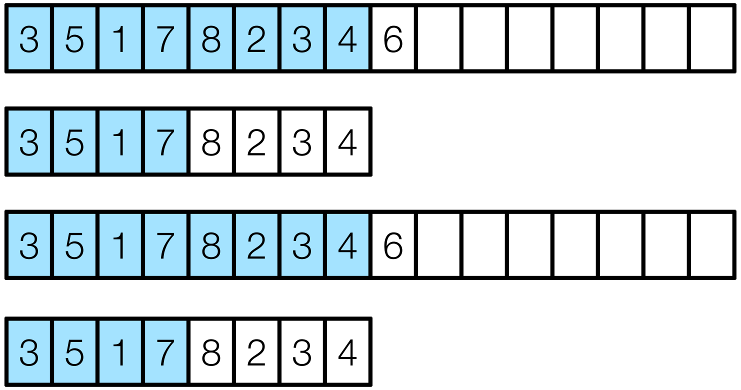

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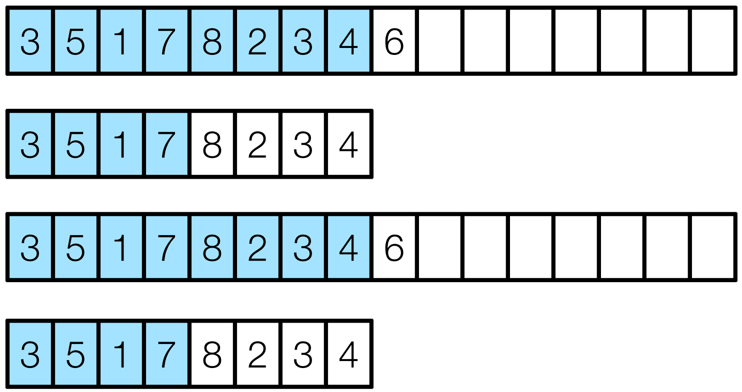

- Consequence. The array is always between 50% and 100% full. **But** risk to use too much time (double or halve every time).

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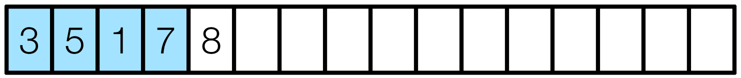

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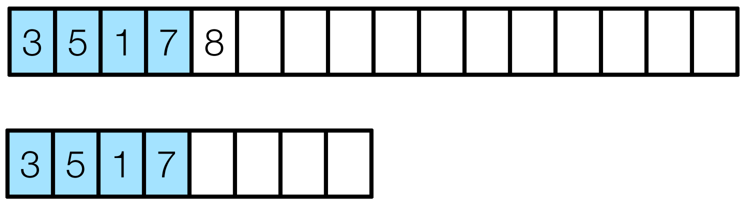

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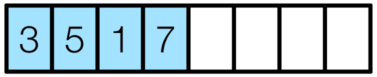

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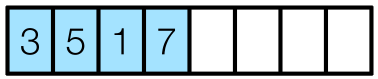

- Consequence. The array is always between 25% and 100% full.

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- Potential method. Define a potential function for the data structure that is initially zero and always non-negative.

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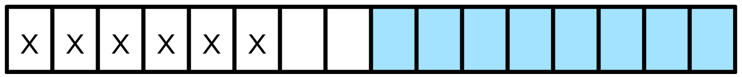

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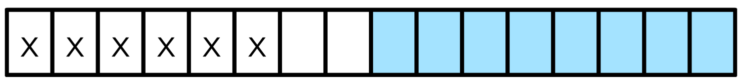

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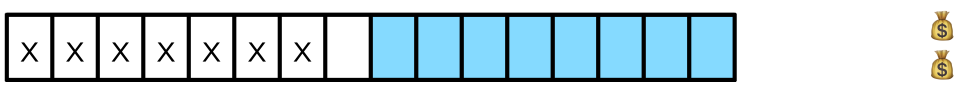

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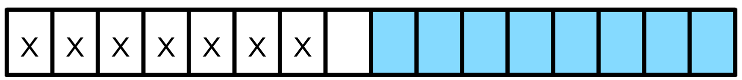

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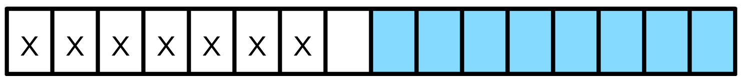

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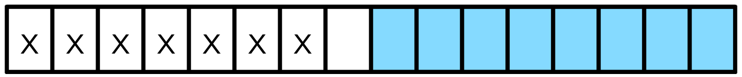

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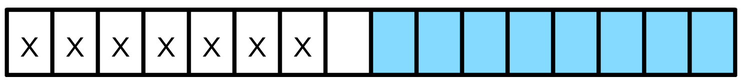

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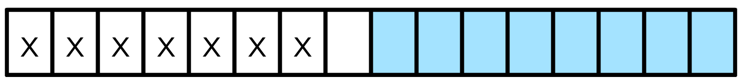

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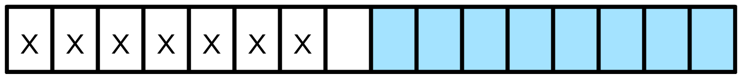

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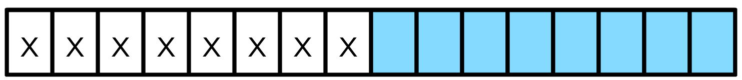

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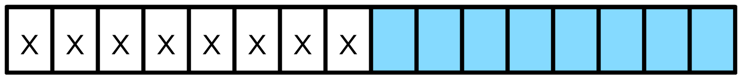

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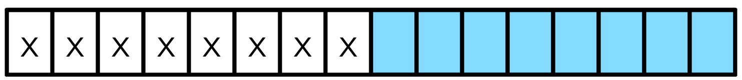

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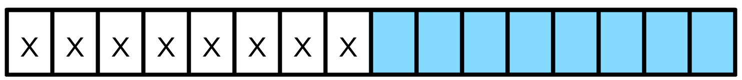

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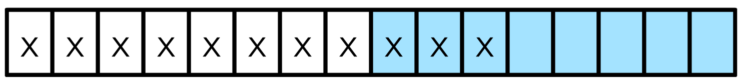
n = 11, L = 16



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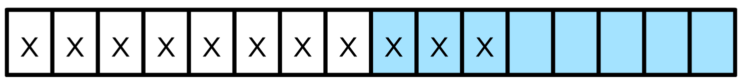
n = 11, L = 16



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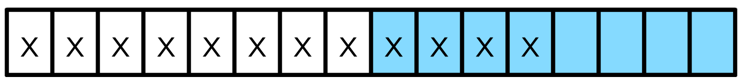

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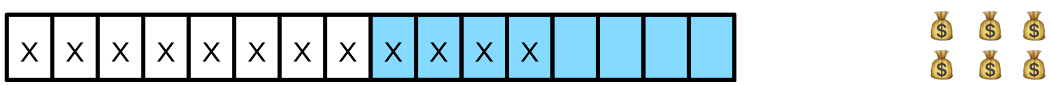
- Inserting when at least half full, but not full:

n = 12, L = 16



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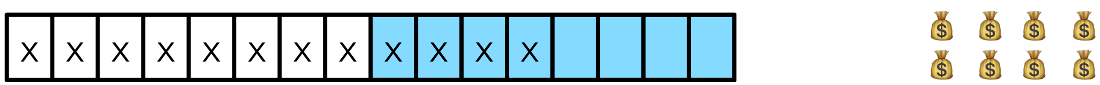
- Inserting when at least half full, but not full:

n = 12, L = 16



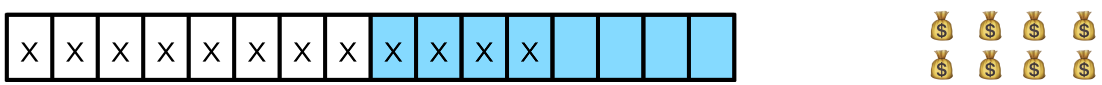
- amortized cost = 1 +

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Inserting when at least half full, but not full:

n = 12, L = 16



- amortized cost = 1 + 💰💰

Dynamic tables

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- Halving. If the table is a quarter full copy the elements to a new array of half the size
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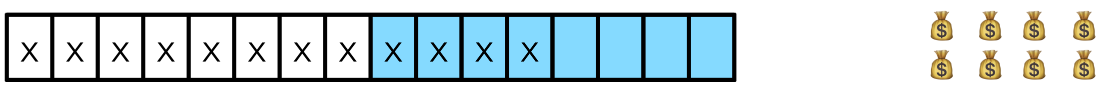
  - L = current array size, n = number of elements in array.
- Inserting when at least half full, but not full:

n = 12, L = 16



- amortized cost = 1 +  = 3

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.
- Inserting in full table and doubling
- amortized cost = 9 +

Dynamic tables

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  - L = current array size, n = number of elements in array.

- Inserting in full table and doubling

n = 8, L = 8

x	x	x	x	x	x	x	x

- amortized cost = 9 +

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
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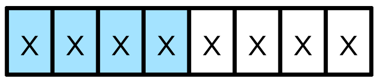
  - $$\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$$
  - L = current array size, n = number of elements in array.

- Inserting in full table and doubling

n = 8, L = 8



- amortized cost = 9 +

Dynamic tables

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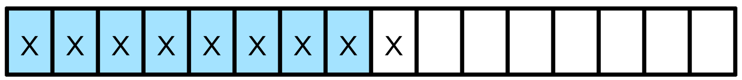
  - L = current array size, n = number of elements in array.

- Inserting in full table and doubling

n = 9, L = 16



- amortized cost = 9 +

Dynamic tables

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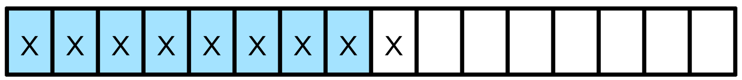
- Inserting in full table and doubling

n = 9, L = 16



- amortized cost = 9 +

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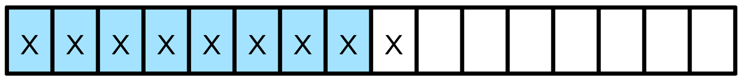
- Inserting in full table and doubling

n = 9, L = 16



- amortized cost = 9 + - 

Dynamic tables

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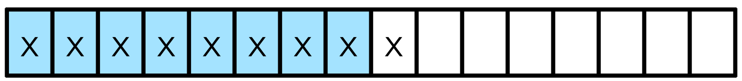
- Inserting in full table and doubling

n = 9, L = 16



- amortized cost = 9 + -  = 3

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.
- Deleting in a quarter full table and halving
- amortized cost = 3 +

Dynamic tables

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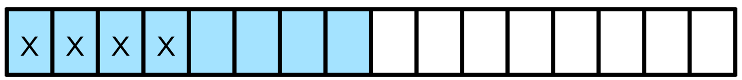
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Deleting in a quarter full table and halving



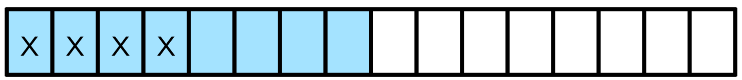
- amortized cost = 3 +

Dynamic tables

- **Doubling.** If the table is **full** (number of elements equal to size of array) copy the elements to a new array of **double** size.
- **Halving.** If the table is a **quarter** full copy the elements to a new array of **half** the size
- **Potential function.**
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Deleting in a quarter full table and halving

n = 4, L = 16



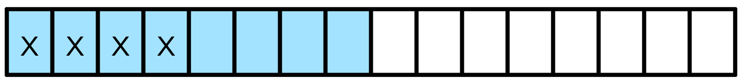
- amortized cost = 3 +

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Deleting in a quarter full table and halving

n = 4, L = 16



- amortized cost = 3 +

Dynamic tables

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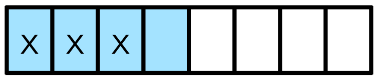
- Potential function.
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Deleting in a quarter full table and halving

n = 3, L = 8



- amortized cost = 3 +

Dynamic tables

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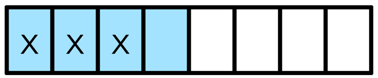
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Deleting in a quarter full table and halving

n = 3, L = 8



- amortized cost = 3 +

Dynamic tables

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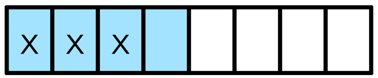
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Deleting in a quarter full table and halving

n = 3, L = 8

💰



- amortized cost = 3 + - 💰 💰 💰

Dynamic tables

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- Halving. If the table is a quarter full copy the elements to a new array of half the size
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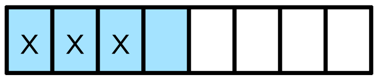
  - L = current array size, n = number of elements in array.

- Deleting in a quarter full table and halving

n = 3, L = 8

💰



- amortized cost = 3 + - 💰 💰 💰 = 0

Dynamic tables

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- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.
- Deleting when more than half full (still half full after):
- amortized cost = 1 +

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
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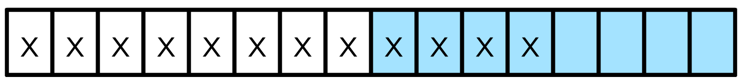
  - L = current array size, n = number of elements in array.

- Deleting when more than half full (still half full after): n = 12, L = 16



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Dynamic tables

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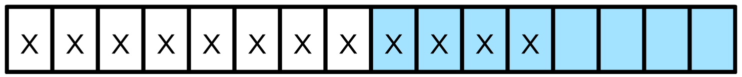
  - L = current array size, n = number of elements in array.

- Deleting when more than half full (still half full after): n = 12, L = 16

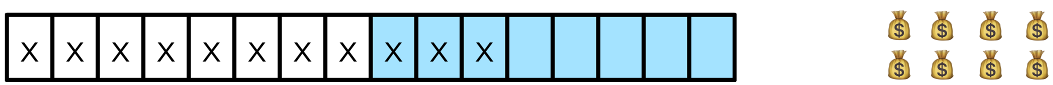


- amortized cost = 1 +

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Deleting when more than half full (still half full after): n = 11, L = 16

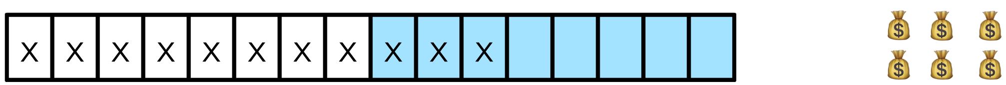


- amortized cost = 1 +

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Deleting when more than half full (still half full after): n = 11, L = 16

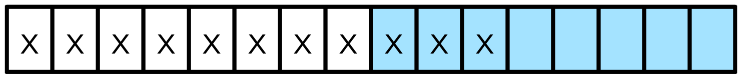


- amortized cost = 1 +

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
    - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
    - L = current array size, n = number of elements in array.

- Deleting when more than half full (still half full after): n = 11, L = 16



- amortized cost = 1 + - 

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
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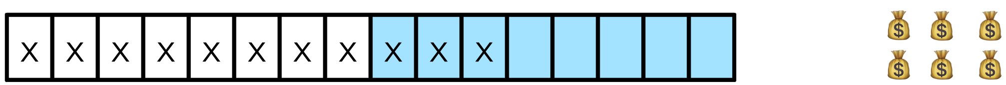
  - L = current array size, n = number of elements in array.

- Deleting when more than half full (still half full after): n = 11, L = 16



- amortized cost = 1 + - = -1

Dynamic tables

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  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.
- Deleting when half full (not half full after):
- amortized cost = 1 +

Dynamic tables

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- Halving. If the table is a quarter full copy the elements to a new array of half the size
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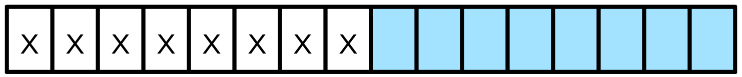
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.
- Deleting when half full (not half full after):

n = 8, L = 16



- amortized cost = 1 +

Dynamic tables

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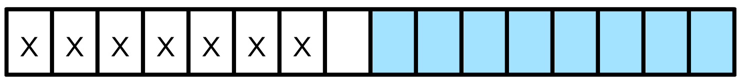
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Deleting when half full (not half full after):

n = 7, L = 16



- amortized cost = 1 +

Dynamic tables

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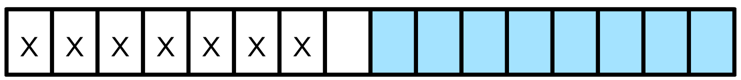
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Deleting when half full (not half full after):

n = 7, L = 16



- amortized cost = 1 +

Dynamic tables

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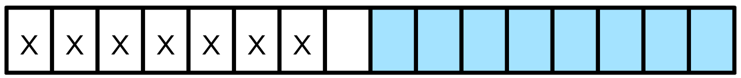
- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
  - $$\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$$
  - L = current array size, n = number of elements in array.

- Deleting when half full (not half full after):

n = 7, L = 16



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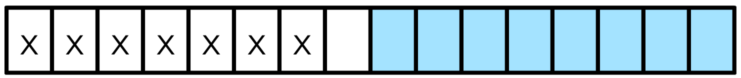
- amortized cost = 1 + 💰

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.

- Deleting when half full (not half full after):

n = 7, L = 16



💰

- amortized cost = 1 + 💰 = 2

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
  - $\Phi(D_i) = \begin{cases} 2n - L & \text{if T at least half full} \\ L/2 - n & \text{if T less than half full} \end{cases}$
  - L = current array size, n = number of elements in array.
- Deleting in when less than half full (but still a quarter full after):
- amortized cost = 1 +

Dynamic tables

- Doubling. If the table is full (number of elements equal to size of array) copy the elements to a new array of double size.
- Halving. If the table is a quarter full copy the elements to a new array of half the size
- Potential function.
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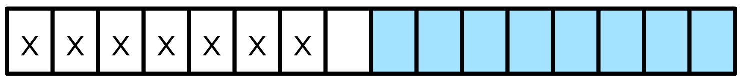
  - L = current array size, n = number of elements in array.

- Deleting in when less than half full (but still a quarter full after):

n = 7, L = 16



- amortized cost = 1 +

Dynamic tables

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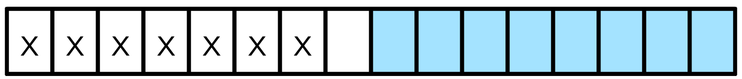

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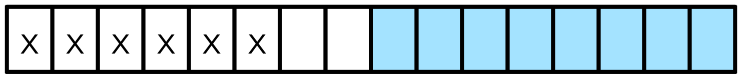

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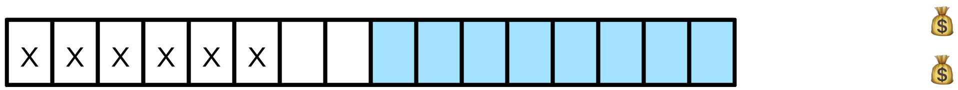

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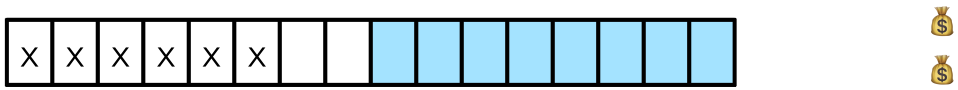

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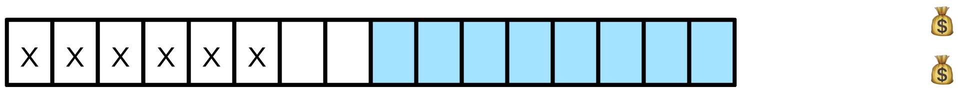


- amortized cost = 1 + 💰 = 2

Potential Method

- Summary:
    1. Pick a potential function, Φ, that will work (art).
    2. Use potential function to bound the amortized cost of the operations you're interested in.
    3. Bound $\Phi(D_0) - \Phi(D_{final})$
- Techniques to find potential functions: if the actual cost of an operation is high, then decrease in potential due to this operation must be large, to keep the amortized cost low.

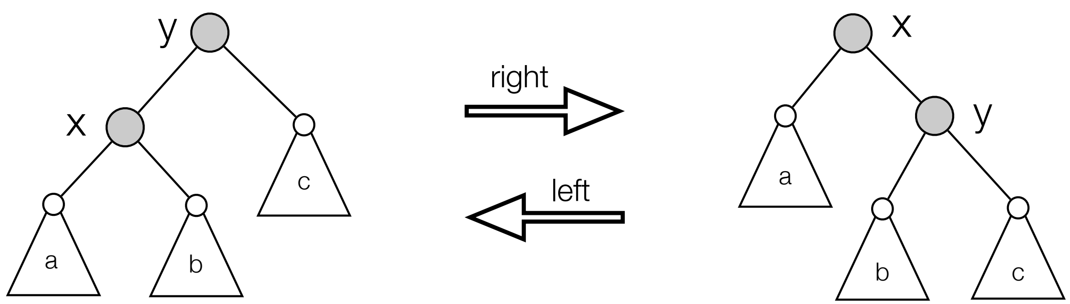
Splay Trees

Splay Trees

- Self-adjusting BST (Sleator-Tarjan 1983).
    - Most frequently accessed nodes are close to the root.
    - Tree reorganizes itself after each operation.
    - After access to a node it is moved to the root by splay operation.
    - Worst case time for insertion, deletion and search is O(n). Amortised time per operation O(log n).
- Operations. Search, predecessor, sucessor, max, min, insert, delete, join.

Splaying

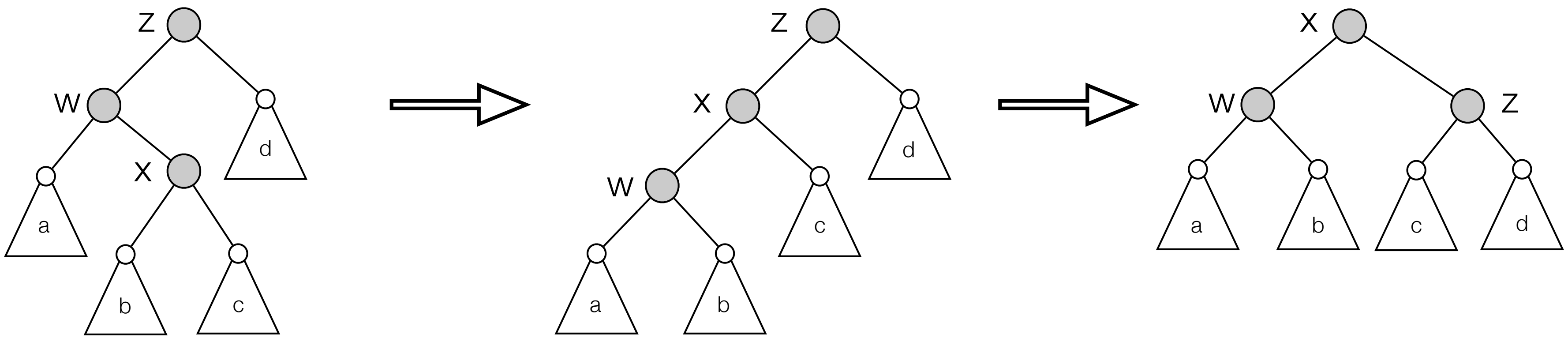
- Splay(x): do following rotations until x is the root. Let y be the parent of x.
    - right (or left): if x has no grandparent.



right rotation at x (and left rotation at y)

Splaying

- Splay(x): do following rotations until x is the root. Let p(x) be the parent of x.
  - right (or left): if x has no grandparent.
  - zig-zag (or zag-zig): if one of x,p(x) is a left child and the other is a right child.



zig-zag at x

Splaying

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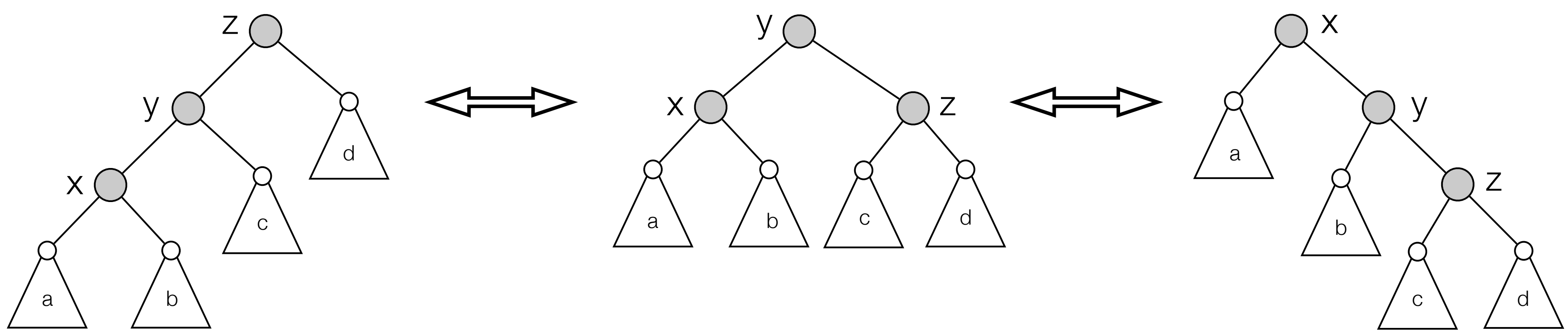
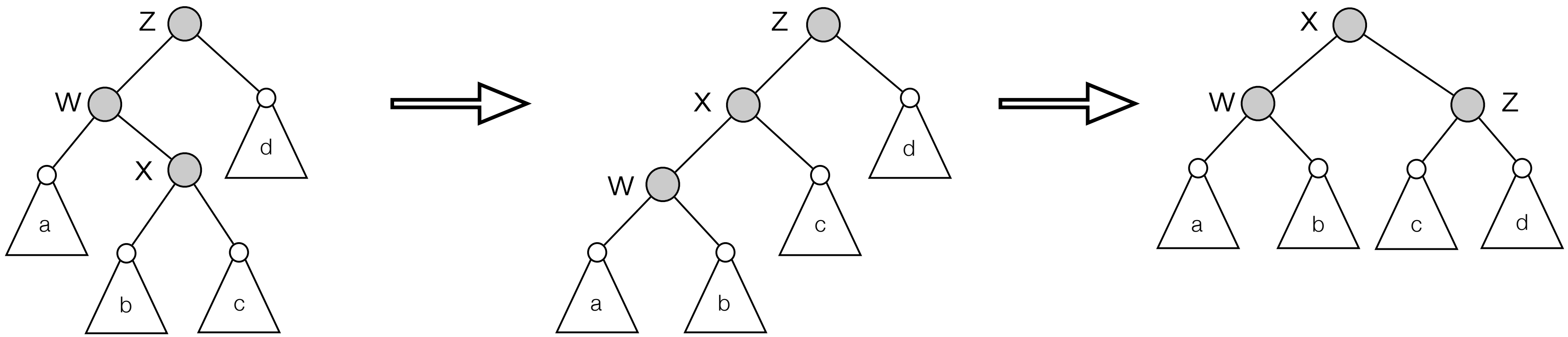
  - right (or left): if x has no grandparent.
  - zig-zag (or zag-zig): if one of x,y is a left child and the other is a right child.
  - roller-coaster: if x and p(x) are either both left children or both right children.



right roller-coaster at x (and left roller-coaster at z)

Splaying

zig-zag at x



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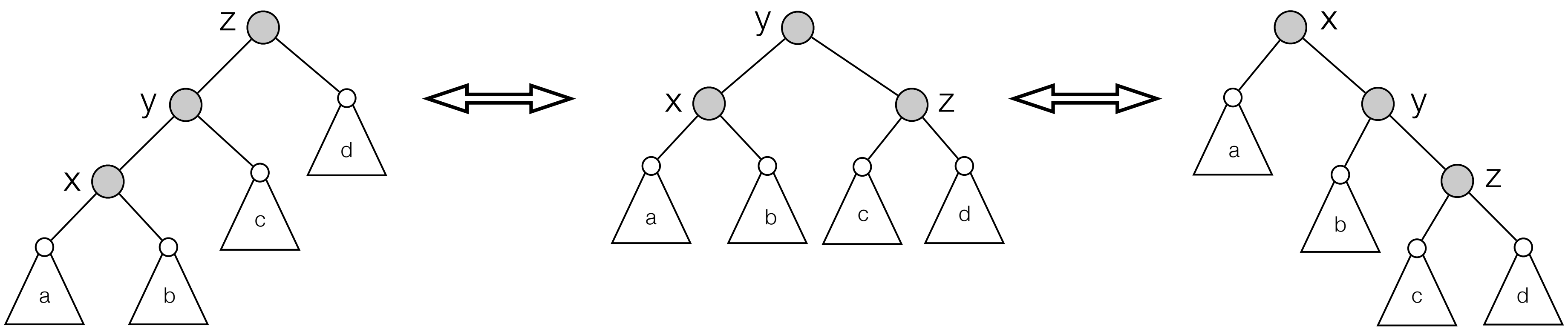
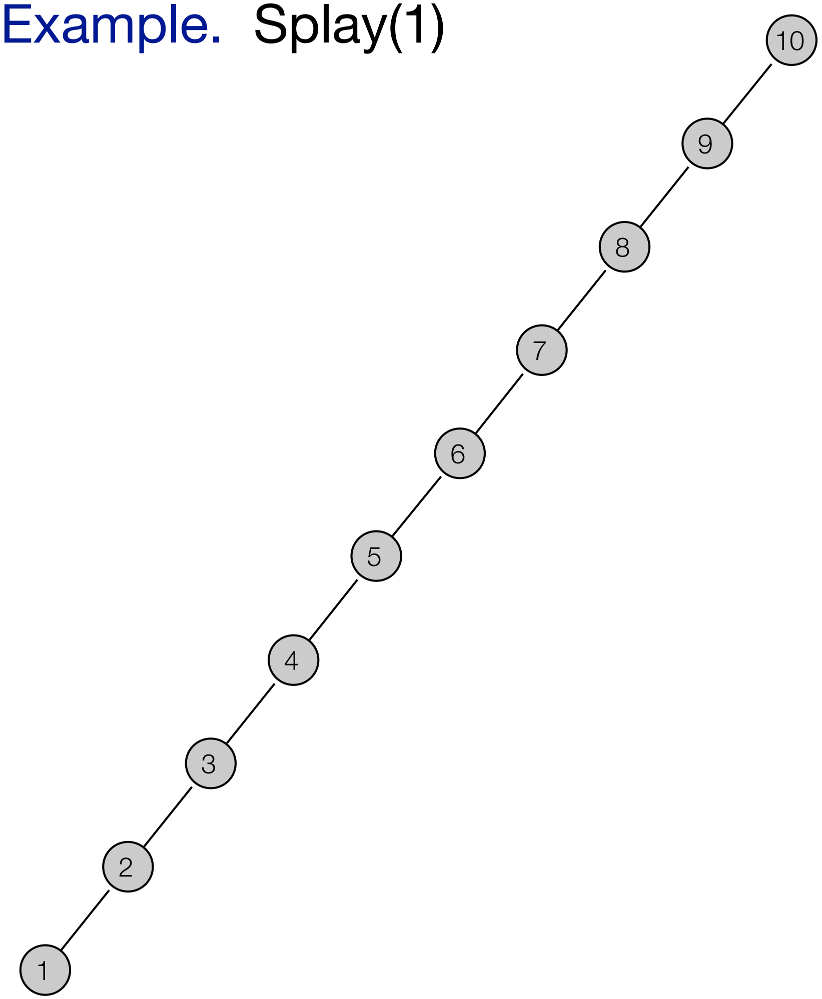


Splay

- **Example.** Splay(1)



right roller-coaster at 1

Splay

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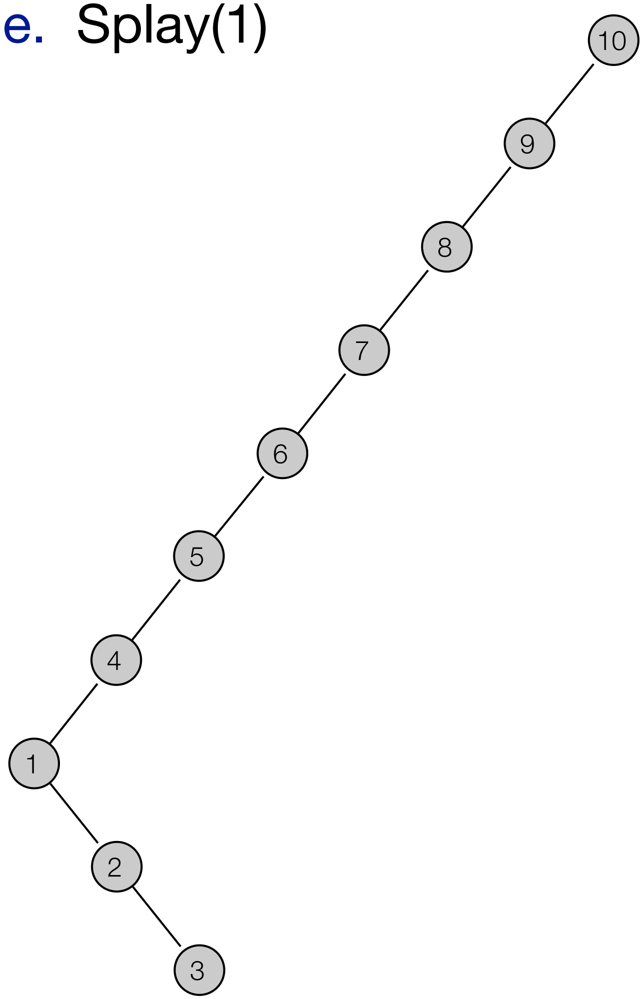

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Splay

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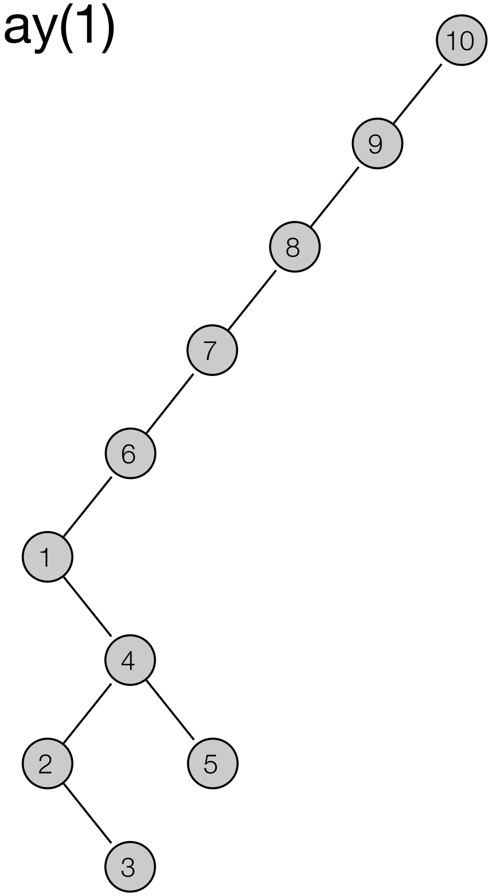

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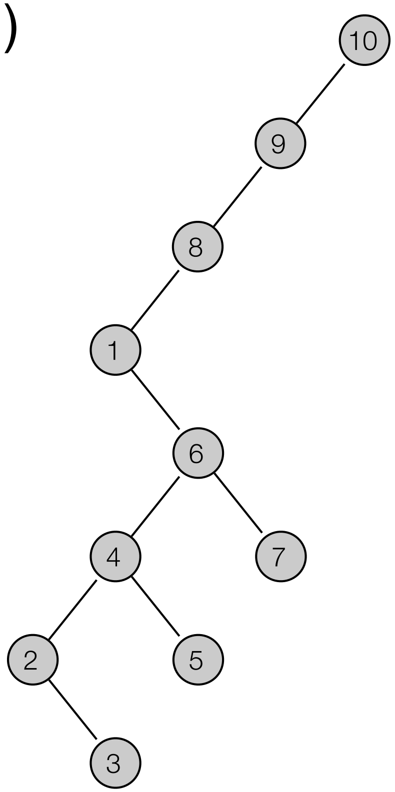

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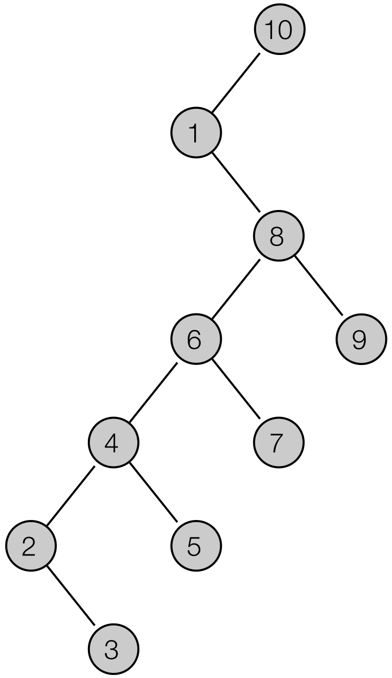

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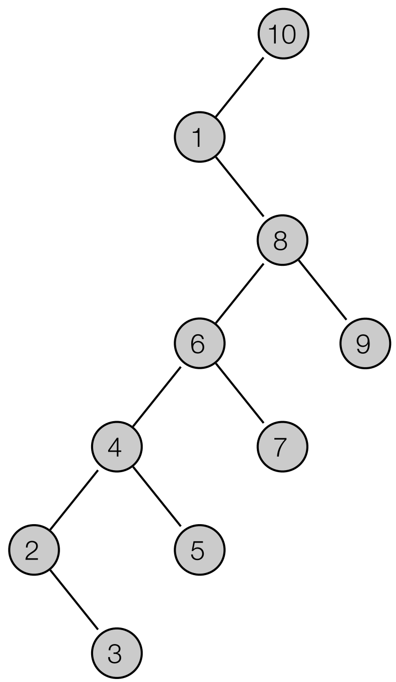

right rotation at 1

Splay

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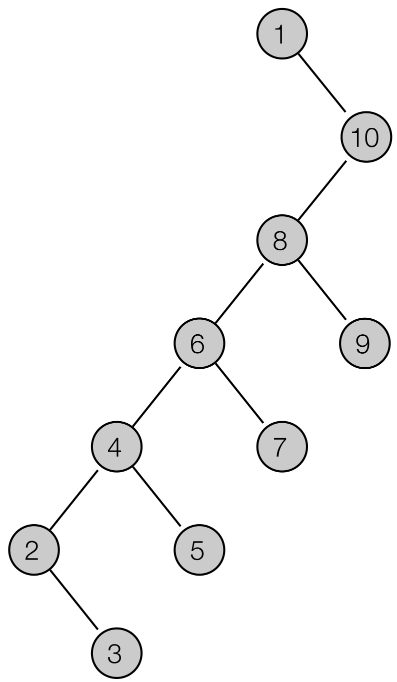
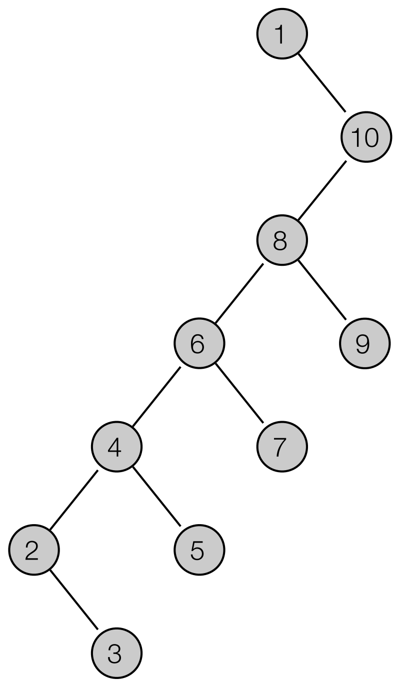


right rotation at 1

Splay

- Example. Splay(3)



zig-zag at 3

Splay

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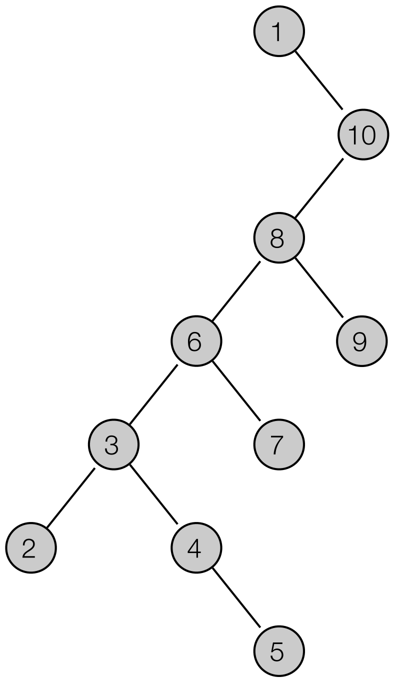

zig-zag at 3

Splay

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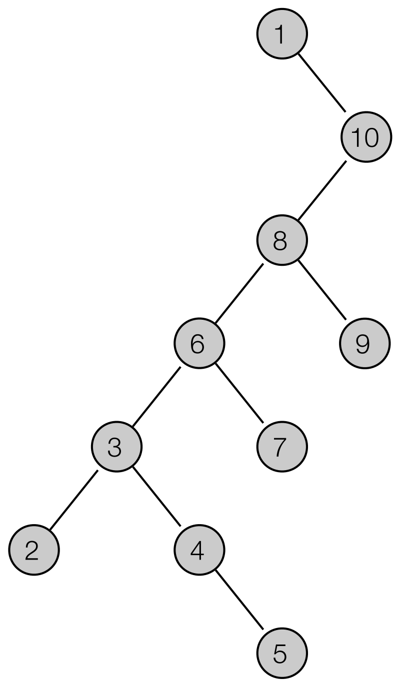

roller-coaster at 3

Splay

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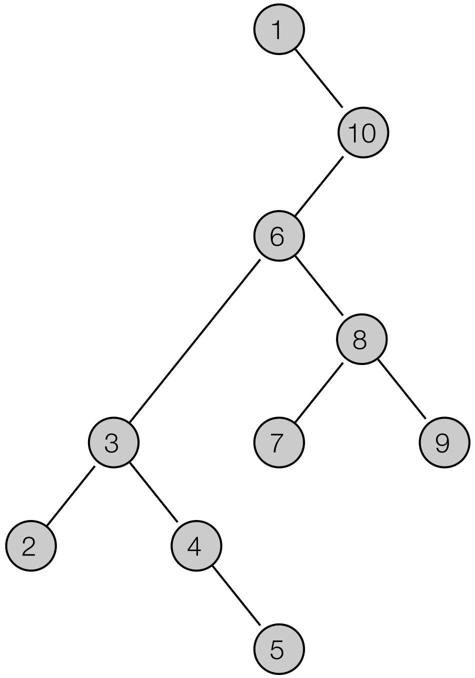

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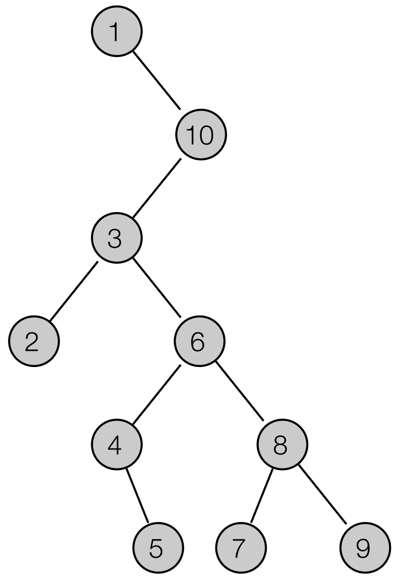

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Splay

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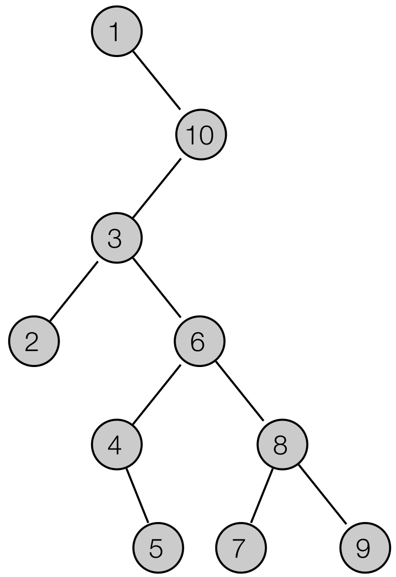

zag-zig at 3

Splay

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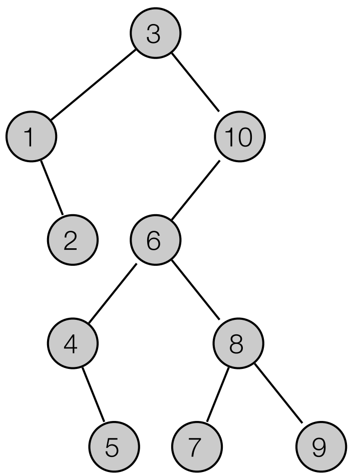

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Splay Trees

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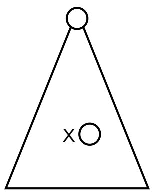

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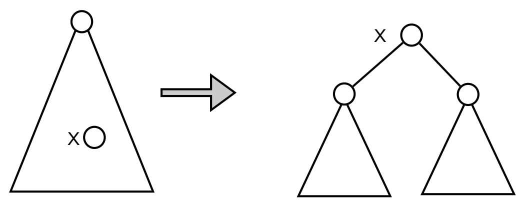

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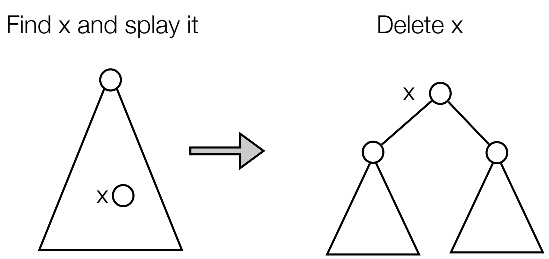

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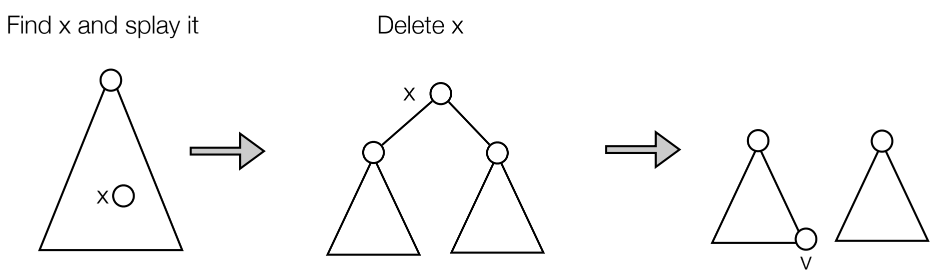

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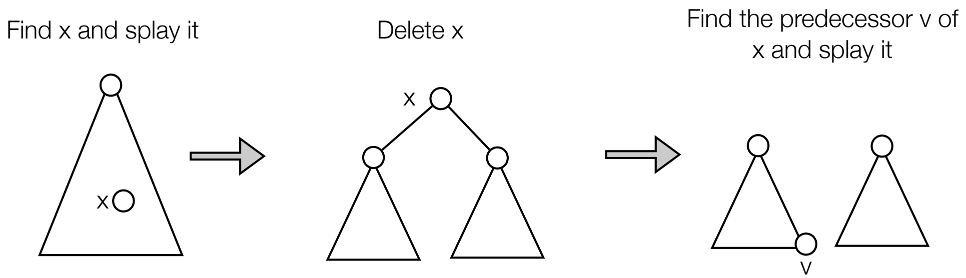

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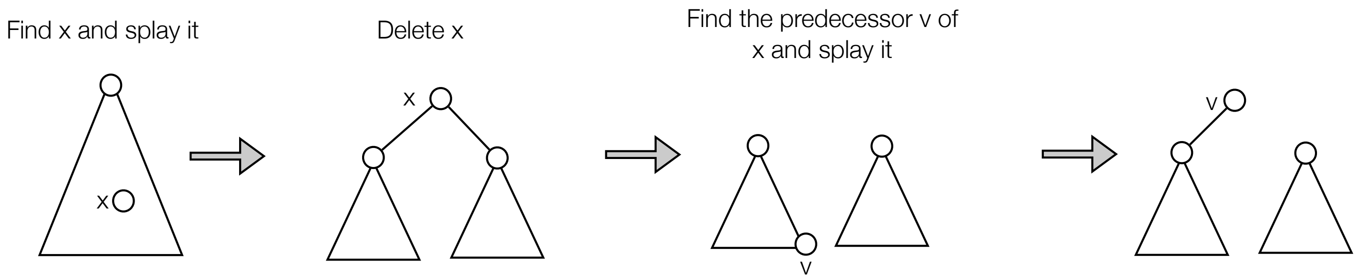

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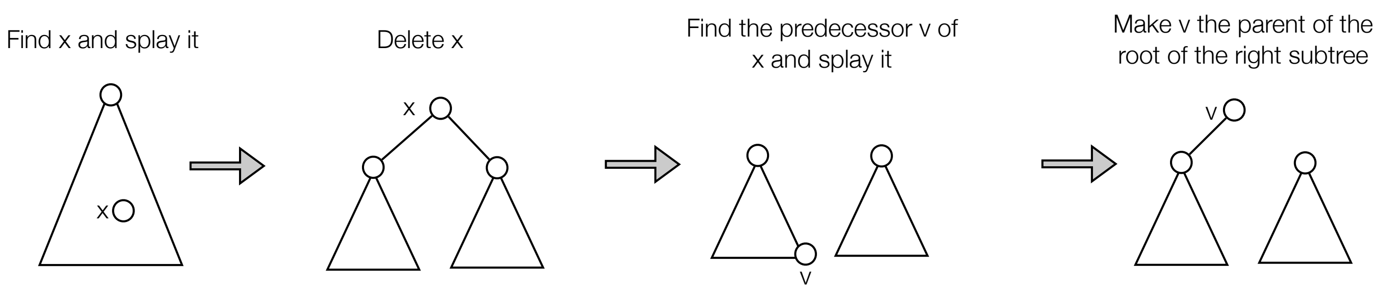

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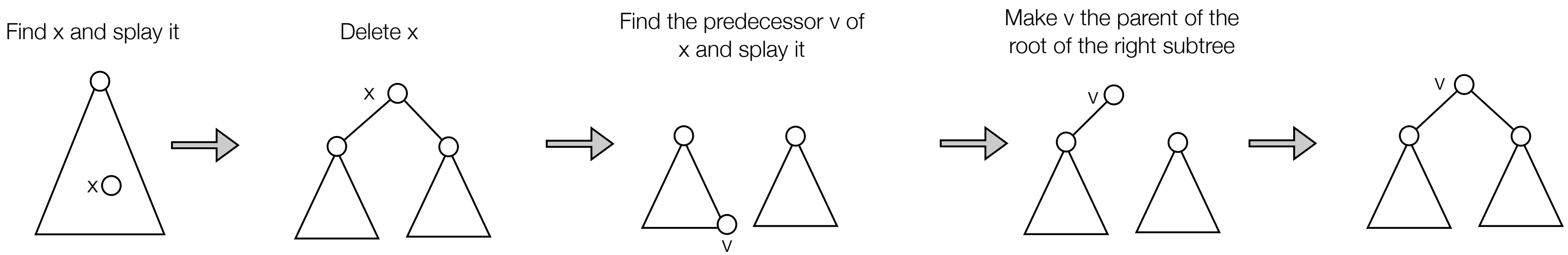
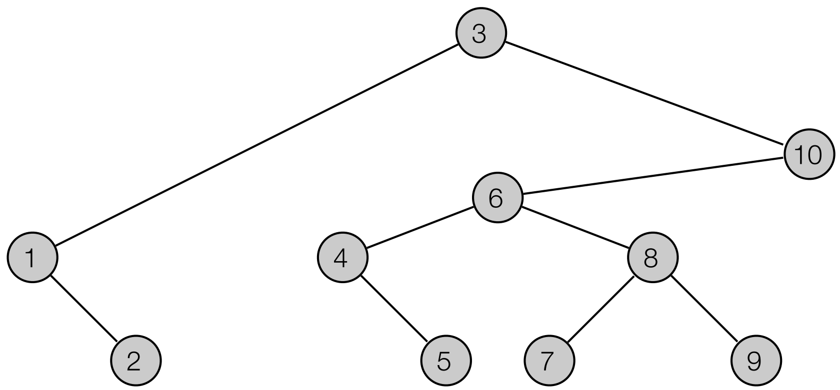


Deletion in Splay Trees

- Delete 6.



splay 6: zag-zig at 6

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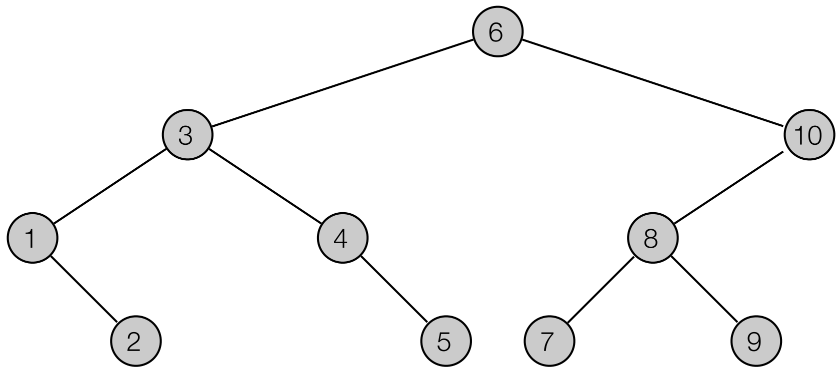

delete 6

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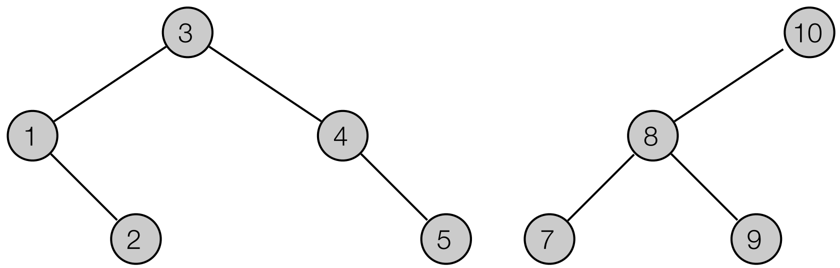

delete 6

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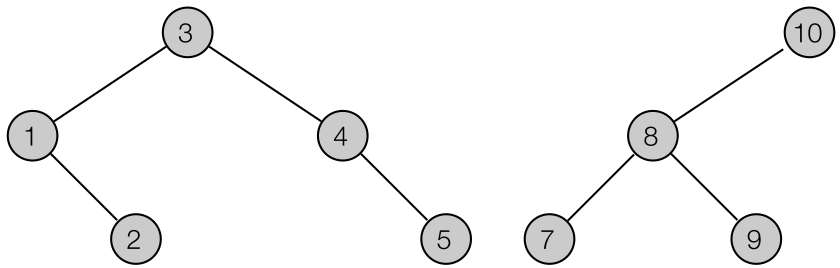

splay 5

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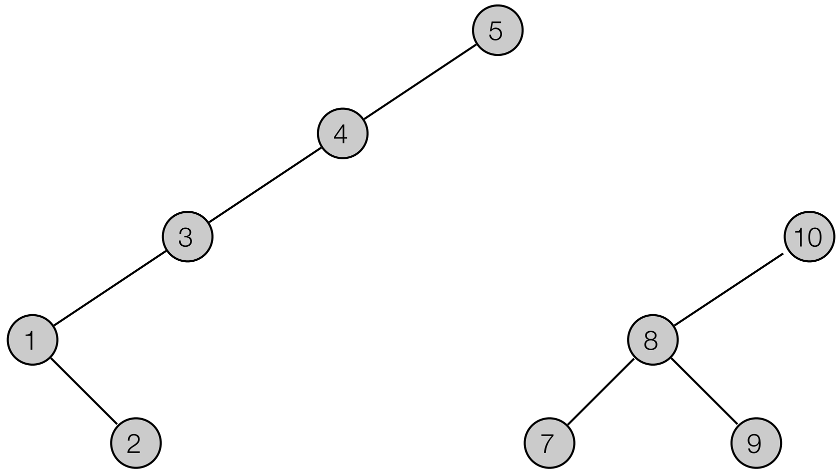

connect

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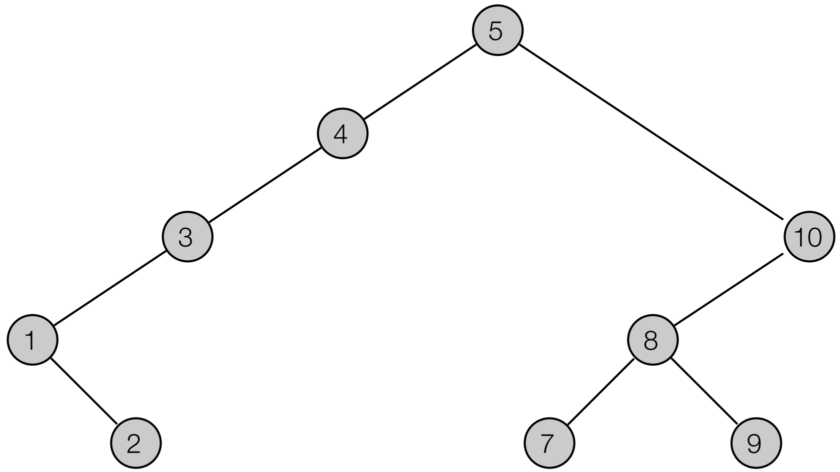

connect

Analysis of splay trees

- Amortized cost of a search, insert, or delete operation is O(log n).
- All costs bounded by splay.

Analysis of splay trees

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- Rank of a node.

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Splay Lemma Proof

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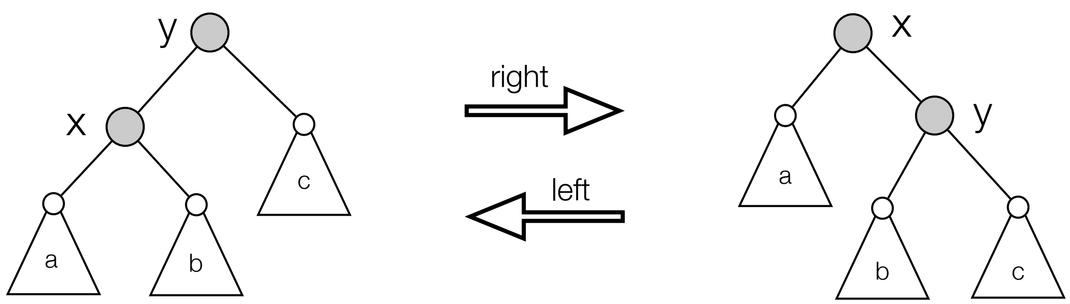
- Splay Lemma. The amortized cost of a splay(v) is at most 1 + 3rank'(v) - 3 rank(v).
- Proof.
    - Assume we have k rotations.
    - Only last one can be a single rotation.

$$\sum_{i=0}^{k} \hat{c}_i \leq \sum_{i=1}^{k-1} \left(r_i(v) - r_{i-1}(v)\right) + (1 + r_k(v) - r_{k-1}(v)) = 1 + r_k(v) - r_0(v)m = O(\lg n)$$

where $r_i(v)$ is the rank of v after the ith rotation.

Rotation Lemma

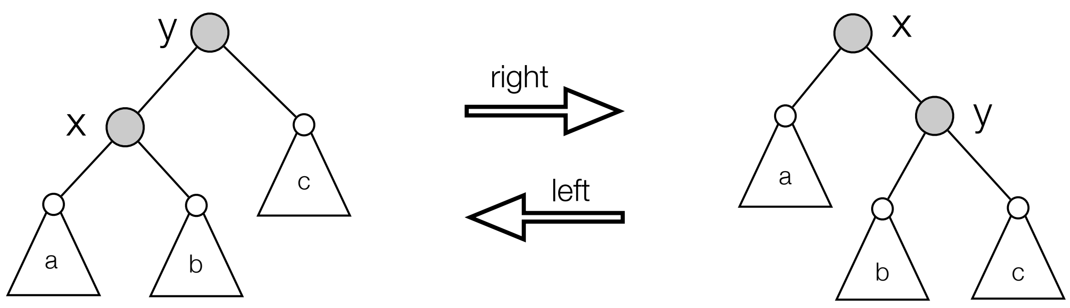
- Proof of rotation lemma: Single rotation.



right rotation at x (and left rotation at y)

Rotation Lemma

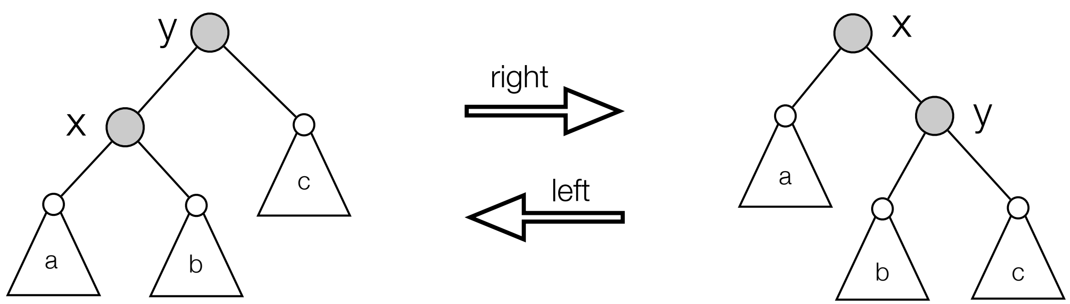
- Proof of rotation lemma: Single rotation.
  - Actual cost: 1



right rotation at x (and left rotation at y)

Rotation Lemma

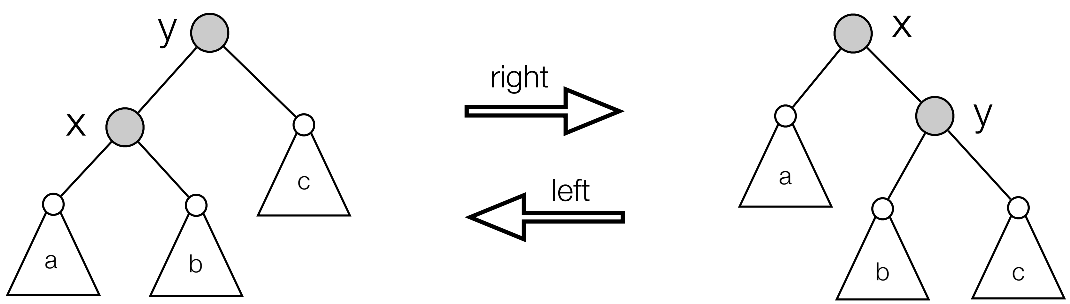
- Proof of rotation lemma: Single rotation.
  - Actual cost: 1
  - Change in potential:



right rotation at x (and left rotation at y)

Rotation Lemma

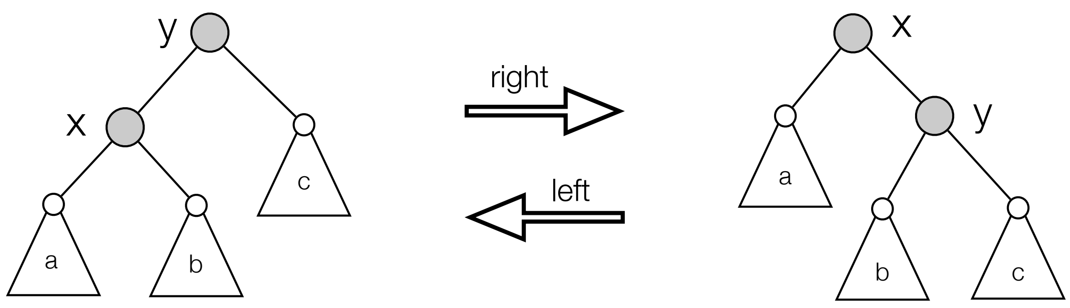
- Proof of rotation lemma: Single rotation.
  - Actual cost: 1
  - Change in potential:
    - Only x and y can change rank.



right rotation at x (and left rotation at y)

Rotation Lemma

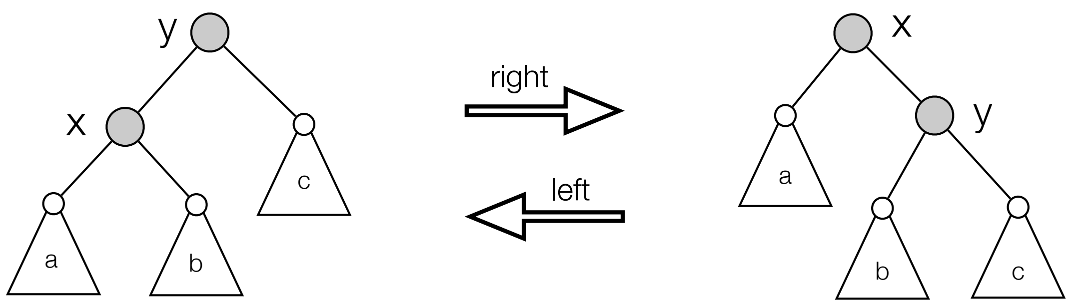
- Proof of rotation lemma: Single rotation.
  - Actual cost: 1
  - Change in potential:
    - Only x and y can change rank.
    - Change in potential at most r'(x) - r(x).



right rotation at x (and left rotation at y)

Rotation Lemma

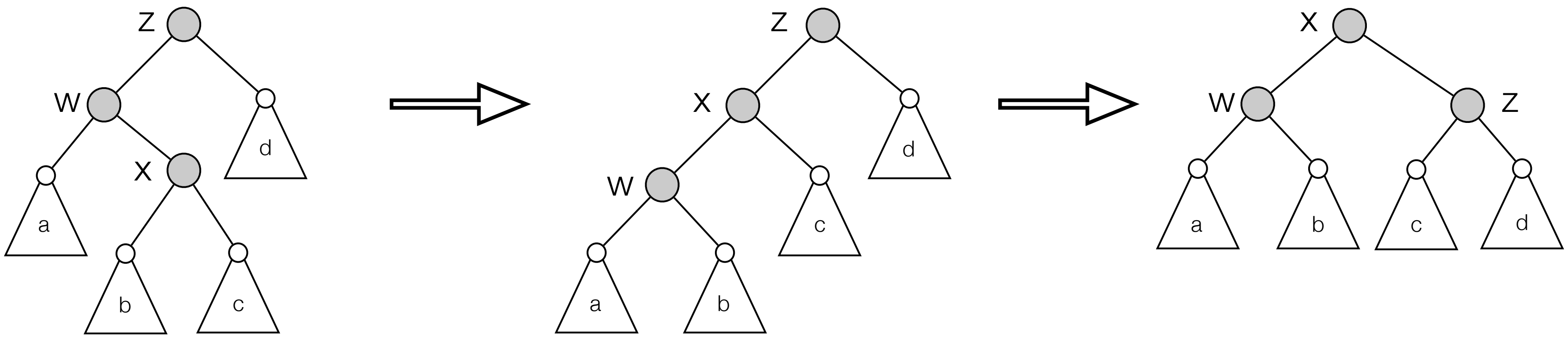
- Proof of rotation lemma: Single rotation.
  - Actual cost: 1
  - Change in potential:
    - Only x and y can change rank.
    - Change in potential at most r'(x) - r(x).
  - Amortized cost ≤ 1 + r'(x) - r(x) ≤ 1 + 3r'(x) - 3r(x).



right rotation at x (and left rotation at y)

Rotation Lemma

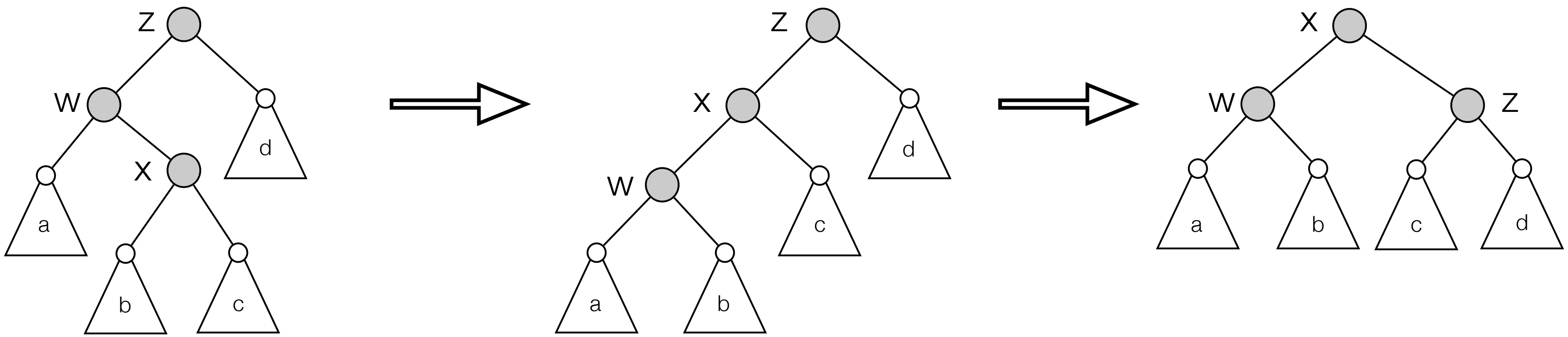
- Proof of rotation lemma: zig-zag.



zig-zag at x

Rotation Lemma

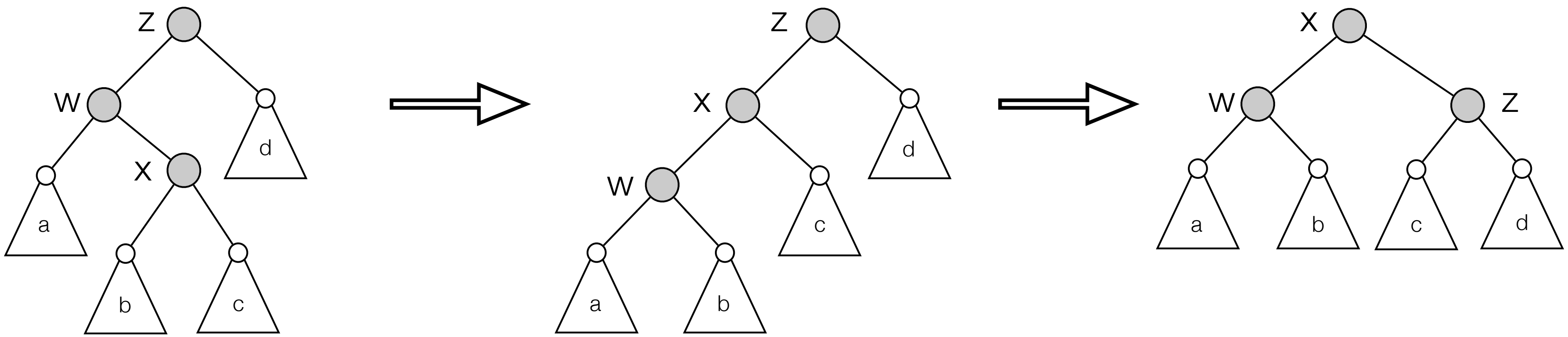
- Proof of rotation lemma: zig-zag.
  - Actual cost: 2



zig-zag at x

Rotation Lemma

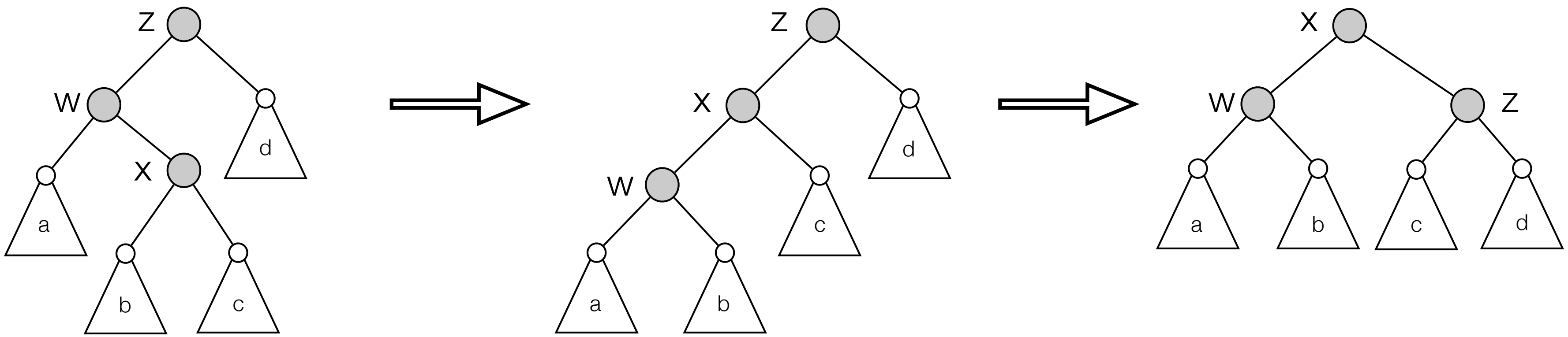
- Proof of rotation lemma: zig-zag.
  - Actual cost: 2
  - Change in potential:



zig-zag at x

Rotation Lemma

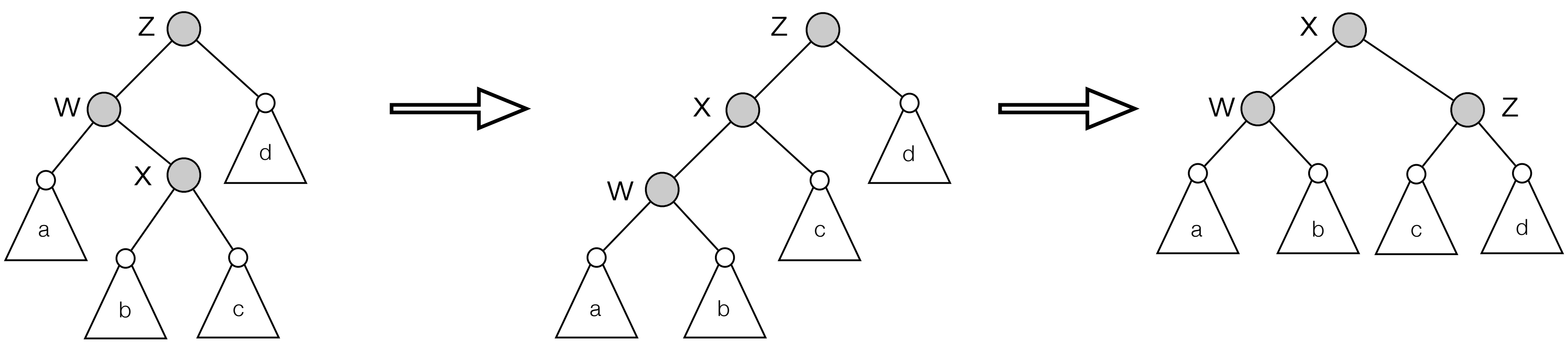
- Proof of rotation lemma: zig-zag.
  - Actual cost: 2
  - Change in potential:
    - Only x, w and z can change rank.



zig-zag at x

Rotation Lemma

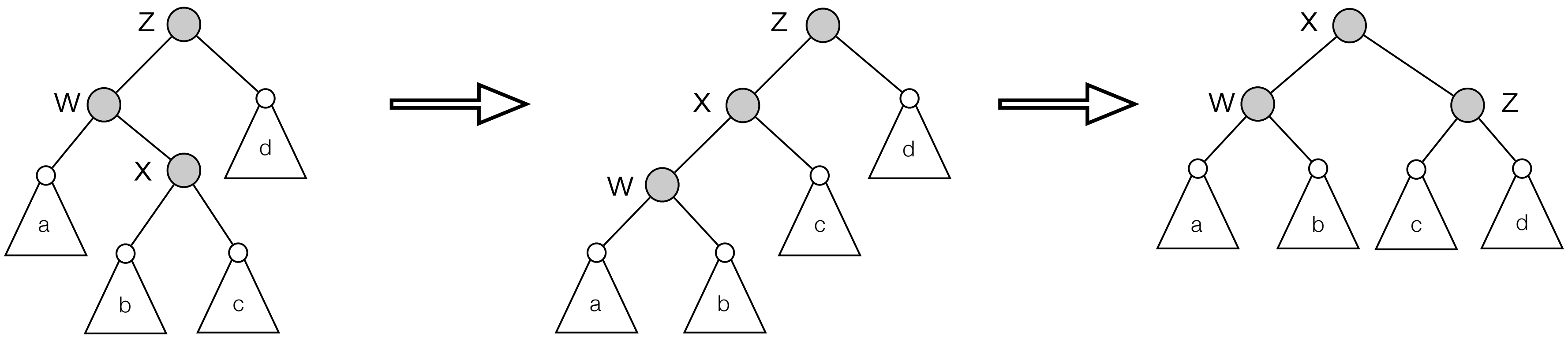
- Proof of rotation lemma: zig-zag.
  - Actual cost: 2
  - Change in potential:
    - Only x, w and z can change rank.
    - Change in potential at most 2r'(x) - 2r(x) - 2.



zig-zag at x

Rotation Lemma

- Proof of rotation lemma: zig-zag.
  - Actual cost: 2
  - Change in potential:
    - Only x, w and z can change rank.
    - Change in potential at most $2r'(x) - 2r(x) - 2$.
  - Amortized cost: $\leq 2 + 2r'(x) - 2r(x) - 2 \leq 2r'(x) - 2r(x) \leq 3r'(x) - 3r(x)$.



zig-zag at x