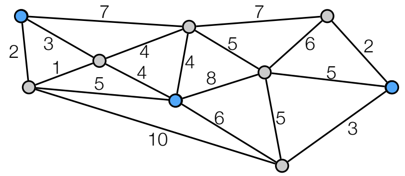
k-center

The k-center problem

- **Input**. An integer k and a complete, undirected graph G=(V,E), with distance d(i,j) between each pair of vertices i,j ∈ V.
- d is a metric:
  - dist(i,i) = 0
  - dist(i,j) = dist(j,i)
  - dist(i,l) ≤ dist(i,j) + dist(j,l)
- **Goal.** Choose a set S ⊆ V , |S| = k, of k centers so as to minimize the maximum distance of a vertex to its closest center.

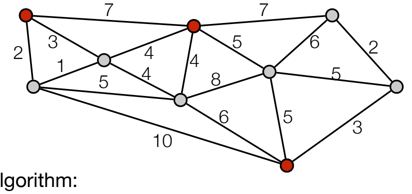
$$S = \text{argmin}_{S\subseteq V, |S|=k} \max_{i\in V} \text{dist}(i,S)$$

- **Covering radius.** Maximum distance of a vertex to its closest center.



k-center: Greedy algorithm

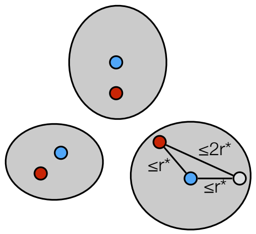
- **Greedy algorithm.**
  - Pick arbitrary i in V.
  - Set S = {i}
  - while |S| < k do
    - Find vertex j farthest away from any cluster center in S
    - Add j to S



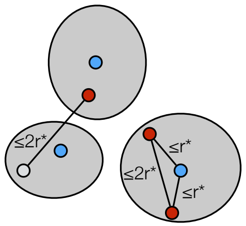
- Greedy is a 2-approximation algorithm:
  - polynomial time ✓
  - valid solution ✓
  - factor 2

k-center: analysis greedy algorithm

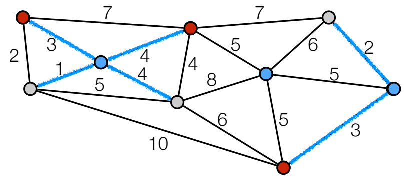
- r* optimal radius.
- Show all vertices within distance 2r* from a center.
- Consider optimal clusters. 2 cases.
  - Algorithm picked one center in each optimal cluster
    - distance from any vertex to its closest center ≤ 2r* (triangle inequality)



  - Some optimal cluster does not have a center.
    - Some cluster have more than one center.
    - distance between these two centers ≤ 2r*.
    - when second center in same cluster picked it was the vertex farthest away from any center.
    - distance from any vertex to its closest center at most 2r*.

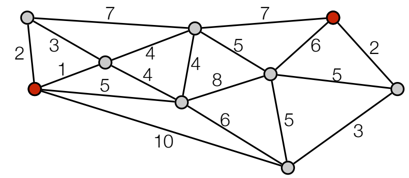


k-center



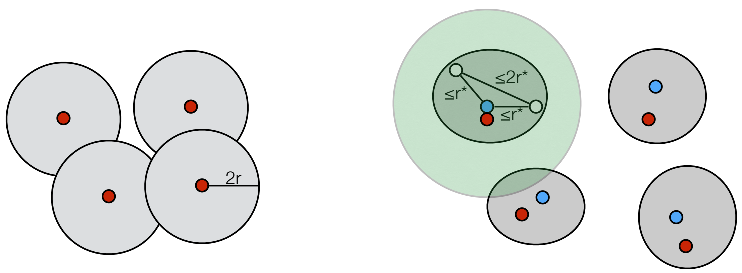
Bottleneck algorithm

- Assume we know the optimum covering radius r.
- Bottleneck algorithm.
  - Set R := V and S := Ø.
  - while R ≠ Ø do
    - Pick arbitrary i in R.
    - Add j to S
    - Remove all vertices with d(j,v) ≤ 2r from R.
- Example: k= 3. r = 4.



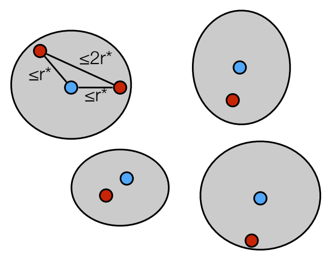
Analysis bottleneck algorithm

- r* optimal radius.
- Covering radius is at most 2r*.
- Show that: We cannot pick more than k centers:
  - We can pick at most one in each optimal cluster:
    - Distance between two nodes in same optimal cluster ≤ 2r.*
    - When we pick a center in a optimal cluster all nodes in same optimal cluster is removed.



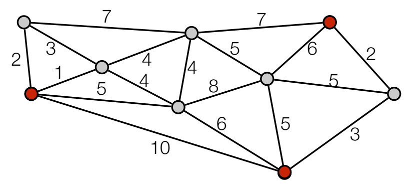
Analysis bottleneck algorithm

- r* optimal radius.
- Can use algorithm to “guess” r* (at most n^2 values).
- If algorithm picked more than k centers then r* > r.
  - If algorithm picked more than k centers then it picked more than one in some optimal cluster.
  - Distance between two nodes in same optimal cluster ≤ 2r.*
  - If more than one in some optimal cluster then 2r < 2r*.



Bottleneck algorithm

- Assume we know the optimum covering radius r.
- Example: k= 3. r* = 4.
- Try with r=2:

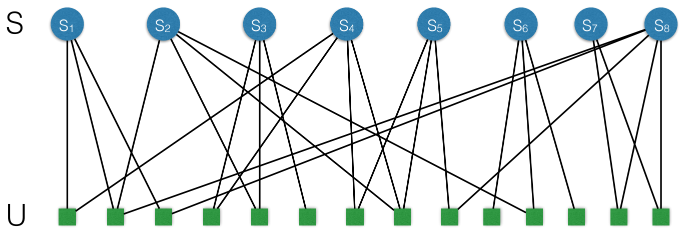


Set cover

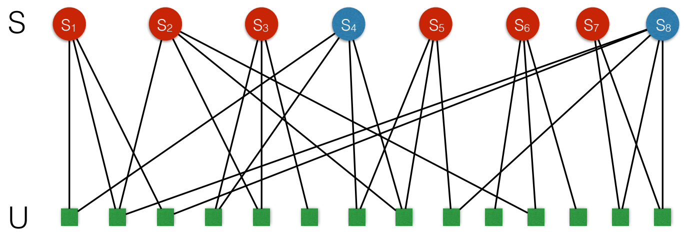
Set cover problem

- Set U of n elements.
- Subsets of U: $S_1,\ldots,S_m$.
- Each set S_i has a weight $w_i \geq 0$.
- Set cover. A collection of subsets C whose union is equal to U.
- Goal. find set cover of minimum weight.

Set Cover



Set Cover



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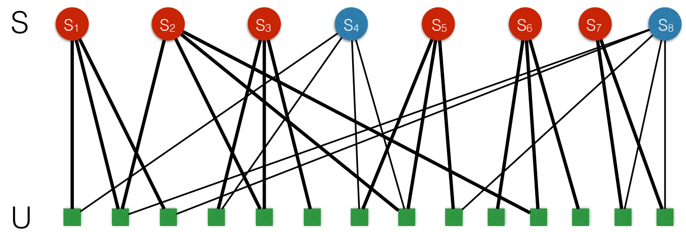

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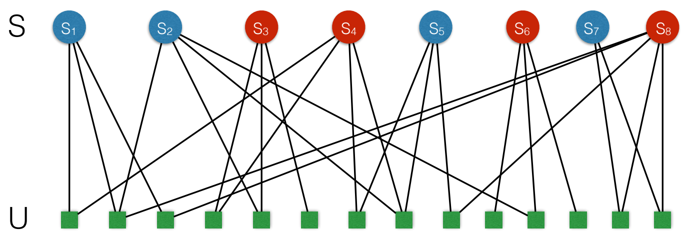

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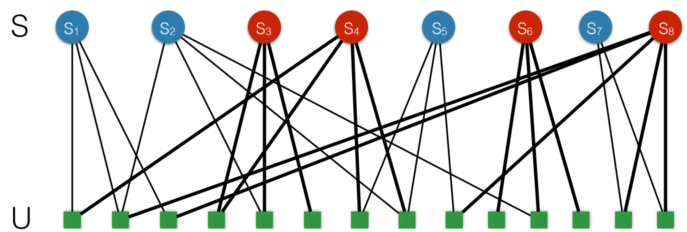
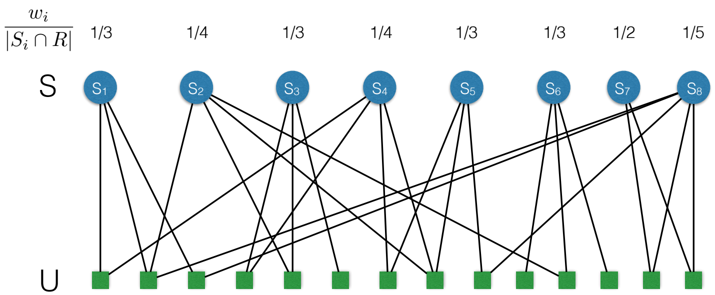
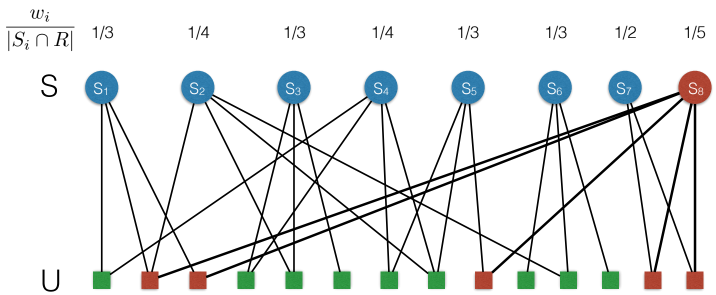


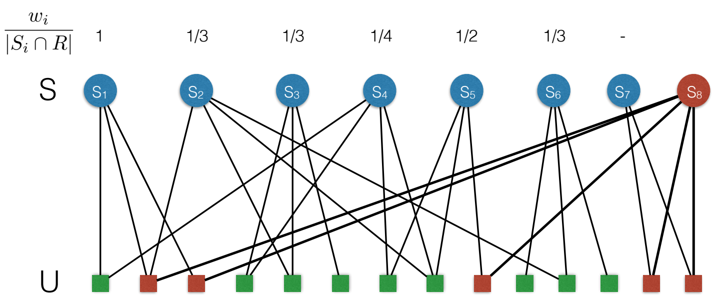
Set Cover: Greedy algorithm



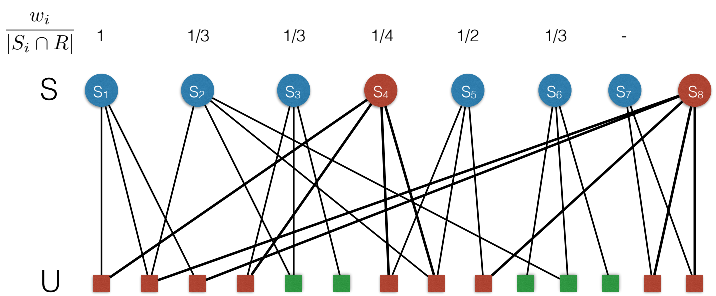
Set Cover: Greedy algorithm



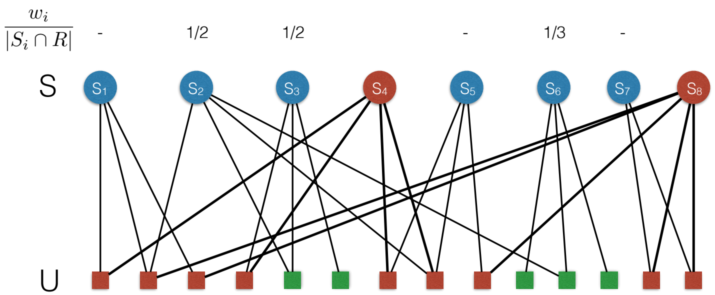
Set Cover: Greedy algorithm



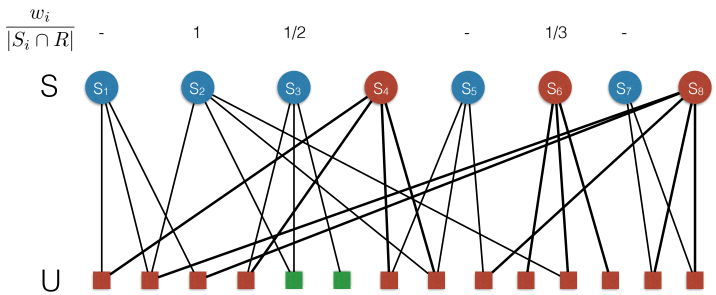
Set Cover: Greedy algorithm



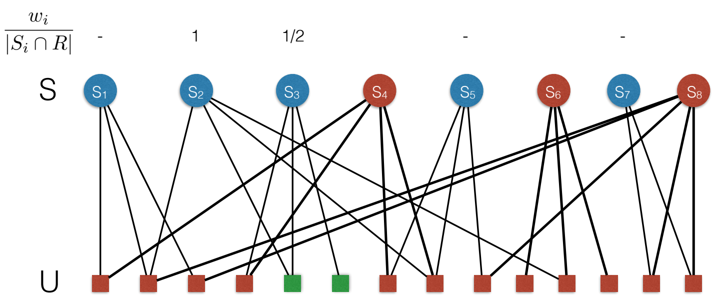
Set Cover: Greedy algorithm



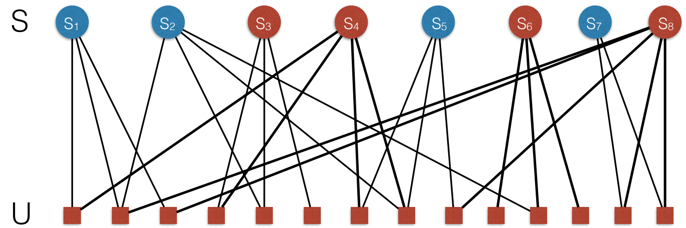
Set Cover: Greedy algorithm



Set Cover: Greedy algorithm



Set Cover: Greedy algorithm

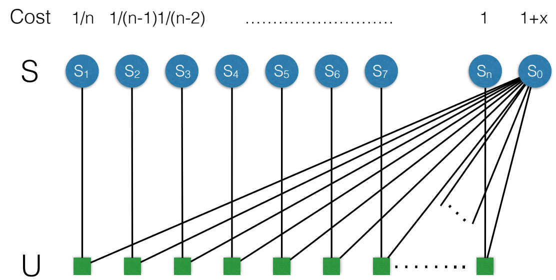


Set cover: greedy algorithm

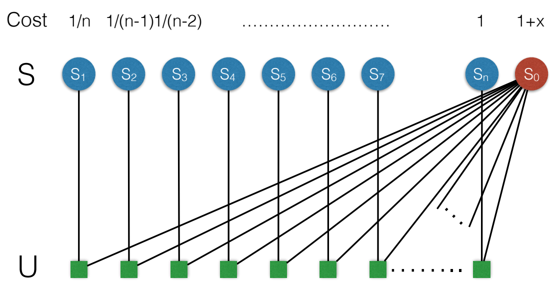
Greedy-set-cover
Set R := U and C := Ø
while R ≠ Ø
Select the set S_i minimizing $\frac{w_i}{|S_i \cap R|}$
Delete the elements from S_i from R.
Add S_i to C.
endwhile
Return C.

- Greedy-set-cover is a n O(log n)-approximation algorithm:
  - polynomial time ✓
  - valid solution ✓
  - factor O(log n)

Set Cover: Greedy algorithm - tight example

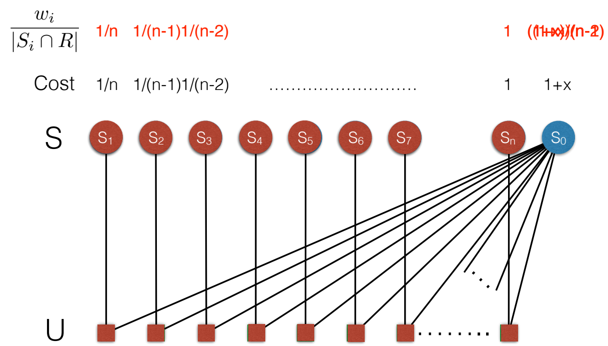


Set Cover: Greedy algorithm - tight example



OPT = 1+x

Set Cover: Greedy algorithm - tight example



OPT = 1+x
Greedy = 1/n + 1/(n-1) + 1/(n-2) …….. = H_n