External Memory

- Computational models
- Shortest path in implicit grid graphs
  - RAM algorithm
  - I/O algorithms
  - Cache-oblivious algorithm

Philip Bille

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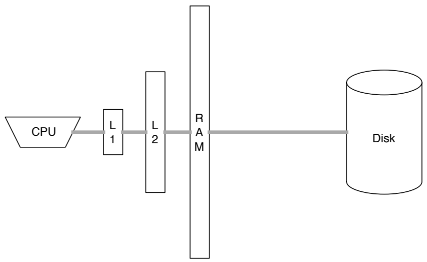
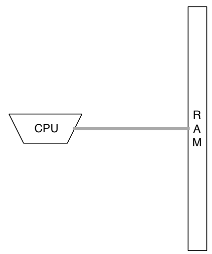
Computational Models



- iPad Air 2.
  - A8X Chip (triple-core ARMv8-A)
  - L1 cache: 64 KB instruction + 64 KB data per core
  - L2 cache: 2 MB
  - L3 cache: 4 MB
  - Memory: 2 GB
  - Disk: 16 GB SSD

Computational Models



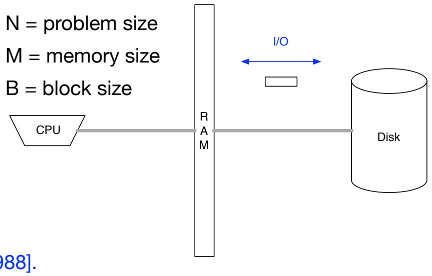
- Word RAM model.
  - Infinite memory of cells.
  - Read/write a cell.
  - Arithmetic and boolean operations (+,-,/,=,<,>,&,|,...)
- Cost.
  - Time complexity = number of operations.

Computational Models

N = problem size

M = memory size

B = block size



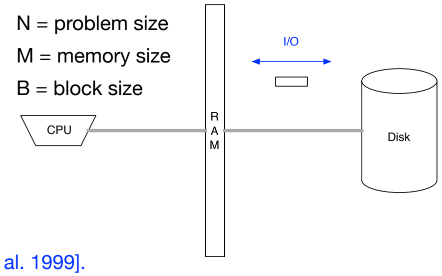
- I/O model [Aggarwal and Vitter 1988].
  - Limited memory + infinite disk
  - I/O operation = read/write consecutive block of B cells between memory and disk.
  - Arithmetic and boolean operations (+,-,/,=,<,>,&,|,...) on cells in memory.
- Cost.
  - I/Os = number of I/O operations.
  - Computation is free (!)

Computational Models

N = problem size

M = memory size

B = block size



- Cache-oblivious model [Frigo et al. 1999].
  - Identical to I/O model except algorithms do not know M and B.
  - Program in RAM model and analyze in I/O model.
  - Assume optimal cache replacement strategy with full associativity.
- Properties.
  - Efficient on one level of cache $\Longrightarrow$ efficient on all levels cache.
  - Portable + self-tunable + simple.

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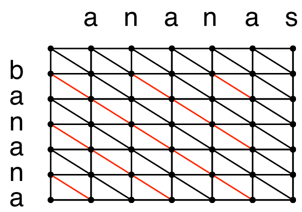
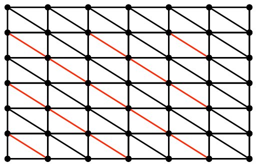
Shortest Paths in Implicit Grid Graphs



- Implicit grid graphs.
  - Let S and T be strings of length n.
  - The implicit grid graph for S and T is A 2D grid of (n+1) x (n+1) nodes.
    - For each node an edge to neighbors to E, S, SE.
    - E and S edges have weight 1.
    - SE edge (i-1,j-1) to (i, j) has weight 0 if S[i] = T[j] and 1 otherwise.

Shortest Paths in Implicit Grid Graphs



- Shortest paths in implicit grid graphs (SPIIG) problem.
  - Input. Strings S and T of length n.
  - Output. Length of shortest path from (0,0) to (n,n).

Shortest Paths in Implicit Grid Graphs

- Applications.
  - Shortest paths in implicit grid graphs is the edit distance problem.
  - With other edge weight functions we get longest common subsequence, sequence alignment, string similarity, approximate string matching, etc.

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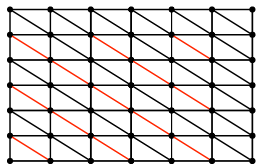
RAM Algorithm



		a	n	a	n	a	s
	0	1	2	3	4	5	6
b	1	1	2	3	4	5	6
a	2	1	2	2	3	4	5
n	3	2	1	2	2	3	4
a	4	3	2	1	2	2	3
n	5	4	3	2	1	2	3
a	6	5	4	3	2	1	2

- How can we solve SPIIGG on a RAM?
- Dynamic programming algorithm.
  - Construct (n+1) x (n+1) matrix.
  - Fill in each entry in O(1) time in left-to-right top-to-bottom order.
- Time. $O(n^2)$
- Space. O(n) (only store current + last row)
- Slightly faster solutions known [MP1980, Myers1999, CLZ2002, BFC2008]

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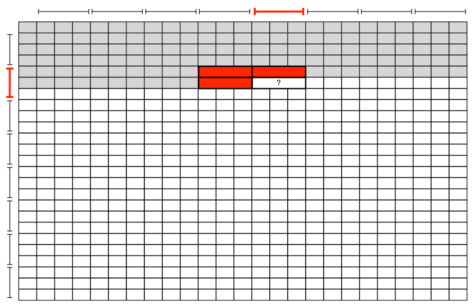
External Memory Algorithms

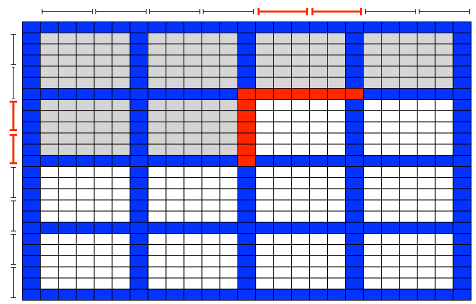
- Goal. Efficient external memory algorithms.
- I/O model.
  - Solution 1. Converted RAM algorithm
  - Solution 2. Table partitioning
- Cache-oblivious model.
  - Solution 3. Recursive table partitioning

Solution 1. Converted RAM Algorithm



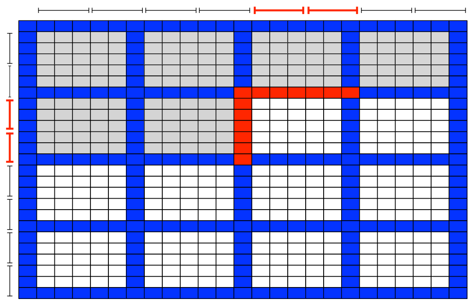
- Strings S and T stored consecutively in n/B blocks on disk.
- Algorithm.
  - Do as RAM algorithm. Read and write blocks as necessary.
- I/Os. $O(n^2/B)$.

Solution 2. Table Partitioning



- Divide into subtables with overlapping boundaries.
- Algorithm. Process subtables from left-to-right, top-to-bottom order. For each subtable:
  - Read corresponding substrings and input boundary into internal memory
  - Fill in subtable using RAM algorithm.
  - Write output boundary to disk.

Solution 2. Table Partitioning

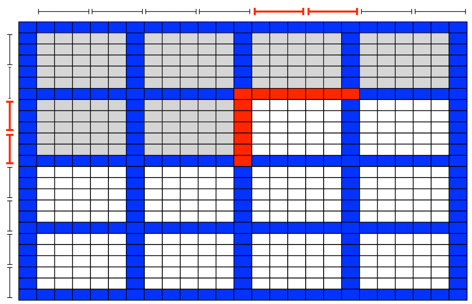


- How to choose subtable size?
- Make subtable dM x dM for d < 1 such that substrings + input boundary + output boundary + space for internal memory algorithm on subtable < M.
- I/Os.
  - Number of subtables = $O(n^2/M^2)$.
  - I/Os per subtable = $O(M/B)$.
  - $\Longrightarrow O(n^2/M^2 \cdot M/B) = O(n^2/MB)$

Solution 2. Table Partitioning

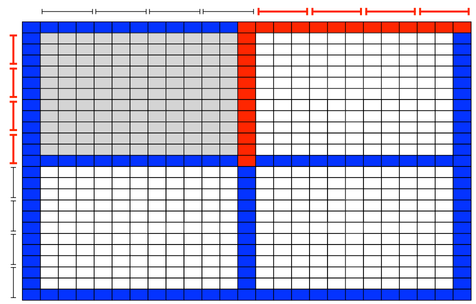
- Theorem. We can solve SPIIGG in the I/O model in
  - $O(n^2/MB + n/B)$ I/Os
  - $O(n^2)$ time
  - $O(n)$ space

Solution 2. Table Partitioning



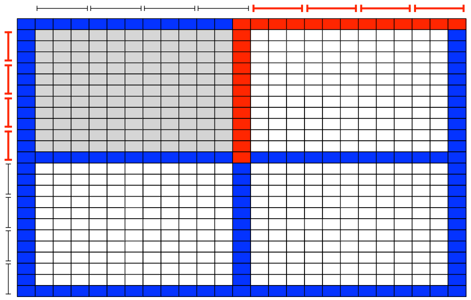
- How can we make solution 2 cache-oblivious?
- Challenge. We cannot use M and B.
- Idea. Use recursion to design algorithm that is good for all M and B.

Solution 3. Recursive Table Partitioning



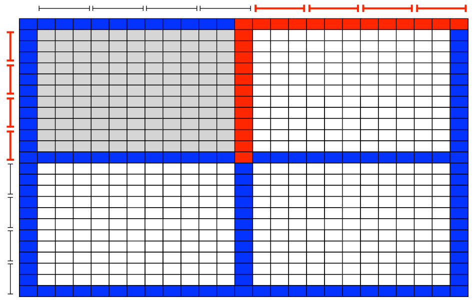
- Algorithm.
- Divide table into 4 quadrants with overlapping boundaries.
- Process quadrants from left-to-right, top-to-bottom order. For each quadrant:
  - Read corresponding substrings and input boundary.
  - Fill in quadrant recursively.
  - Write output boundary.

Solution 3. Recursive Table Partitioning



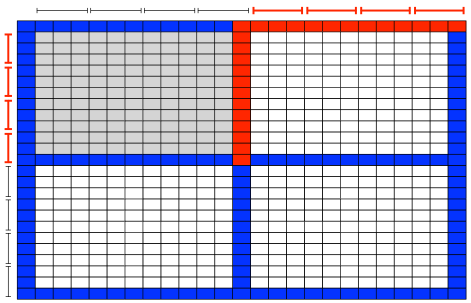
- I/Os.
- Define IO(n) = number of I/Os to process a table of size n x n
  - Case 1: n ≤ dM (substrings + boundaries + computation fit in internal mem)
    - IO(n) = O(n/B)
  - Case 2: n > dM?

Solution 3. Recursive Table Partitioning



- Algorithm.
- Divide table into 4 quadrants with overlapping boundaries. O(1)
- Process quadrants from left-to-right, top-to-bottom order. For each quadrant:
  - Read corresponding substrings and input boundary. O(n/B)
  - Fill in quadrant recursively. 4 · IO(n/2)
  - Write output boundary. O(n/B)

Solution 3. Recursive Table Partitioning



- Case 1 + 2:

$$IO(n) = \begin{cases} O(n/B) & \text{if } n \leq dM \\ 4 \cdot IO(n/2) + O(n/B) & \text{if } n > dM \end{cases}$$

- ⟹ IO(n) = $O(n^2/MB)$

Solution 3. Recursive Table Partitioning

- Theorem. We can solve SPIIGG in the cache-oblivious model in
  - $O(n^2/MB + n/B)$ I/Os
  - $O(n^2)$ time
  - O(n) space

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