Weekplan: Approximation Algorithms II

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References and Reading

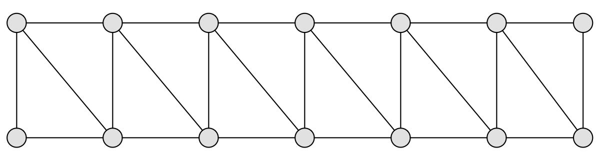
[1] Algorithm Design, Kleinberg and Tardos, Addison-Wesley, section 11.2.

[2] A unified approach to approximation algorithms for bottleneck problems, D. S. Hochbaum and D. B. Shmoys, Journal of the ACM, Volume 33 Issue 3, 1986.

[3] The Design of Approximation Algorithms, Williamson and Shmoys, Cambridge Press, section 2.2.

We expect you to read [1] in detail before the lecture. [3] is alternative reading. [2] provides background on the k-center problem.

1 [w] **k-center** Run both k-center algorithms on the example below with $k = 4$. All edges have length 1.



2 The k-supplier problem The *k-supplier* problem is similar to the k-center problem, but the vertices are partitioned into *suppliers* $F \subseteq V$ and *customers* $C \subseteq V$. The goal is to find k suppliers such that the maximum distance from a customer to a supplier is minimized. Give a 3-approximation algorithm for the k-suppliers problem.

3 Metric k-clustering Give an 2-approximation algorithm for the following problem.

Let $G = (V, E)$ be a complete undirected graph with edge costs satisfying the triangle inequality, and let k be a positive integer. The problem is to partition V into sets $V_1, \dots, V_k$ so as to minimize the costliest edge between two vertices in the same set, i.e., minimize

$$\max_{1 \le i \le k,\, u,v, \in V_i} c(u, v).$$

4 Priority k-center Consider the following variant of the k-center problem, where the vertices have priorities: Each vertex have a priority, and we want to find a set of k centers so that the maximum *prioritized* distance of a vertex to its closest center is minimized. That is, the higher value priority a vertex has, the closer it should be to a center.

Formally, in the *prioritized k-center problem* we are given a complete graph $G = (V, E)$ with a cost function on the edges $d : E \to Q^+$ satisfying the triangle inequality, a priority function on vertices: $p : V \to \mathbb{R}^+$, and a positive integer k. The problem is to find a set of centers $C \subseteq V$ with $|C| \le k$ minimizing

$$r(C) = \max_{v \in V} p(v) \cdot d(v, C),$$

where

$$d(v, C) = \min_{u \in C} d(v, u).$$

The following algorithm for the prioritized k-center problem assumes we know the optimal radius r.

Algorithm 1 Prority-Center (G, r)

Set $S = V$ and $C = \emptyset$.
while $S \neq \emptyset$ **do**
  Select the heaviest vertex $v \in S$ (the vertex with highest priority)
  Set $C = C \cup \{v\}$
  Remove all vertices u from S with $p(u) \cdot d(u, v) \leq 2r$ from v.
end while
Return C

4.1 Assume we know the optimum covering radius r. Let C be the set of centers computed by the algorithm $kCenter(G, r)$, let C^* be the set of optimal centers (each vertex is assigned to its closest center).

(a) Consider an iteration of the algorithm and let v be the vertex chosen in this iteration. Let c^* be the center v is assigned to in the optimal solution. Let $z \in S$ be a vertex assigned to c^* in the optimal solution. Show that $p(z) \cdot d(z, v) \leq 2r$.

(b) Show that at most one vertex from each cluster from C^* belongs to C.

4.2 Prove that Algorithm 1 is a 2-approximation algorithm for the prioritized k-center problem (assuming that we know the optimal covering radius r).

5 Vertex cover A *vertex cover* in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ so that each edge has at least one end in S. In the *cardinality vertex cover problem* the goal is to find a vertex cover of the input graph $G = (V, E)$ of minimum size.

A *matching* in a graph $G = (V, E)$ is subset of edges $M \subseteq E$ so that no two edges of M hare an endpoint. A *maximal matching* is a matching that is maximal under inclusion. That is, adding any edge from E to the maximal matching will cause two edges in M to share an endpoint.

5.1 Show that a maximal matching can be computed in polynomial time.

5.2 Show that the size of a maximal matching in a graph G is a lower bound on the size of the minimum vertex cover in G.

5.3 Give a 2-approximation algorithm for cardinality vertex cover.