Approximation Algorithms II

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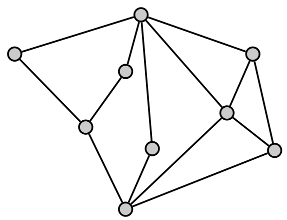
Today

- Hardness of approximation of TSP.
- k-center problem

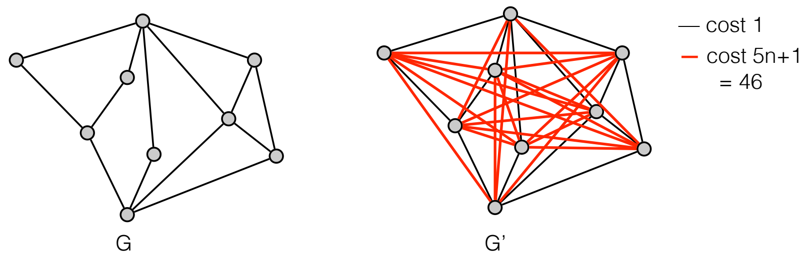
TSP: Inapproximability

- There is no α-approximation algorithm for the TSP for unless P=NP.

- *Hamiltonian cycle.* Given G=(V,E). Is there a cycle visiting every vertex exactly once?

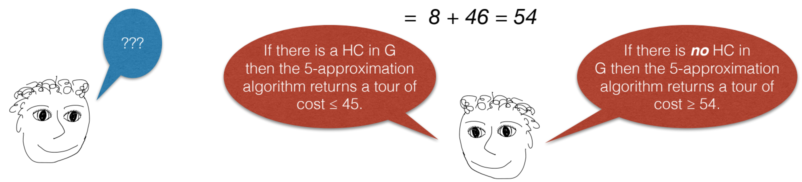


TSP: Inapproximability

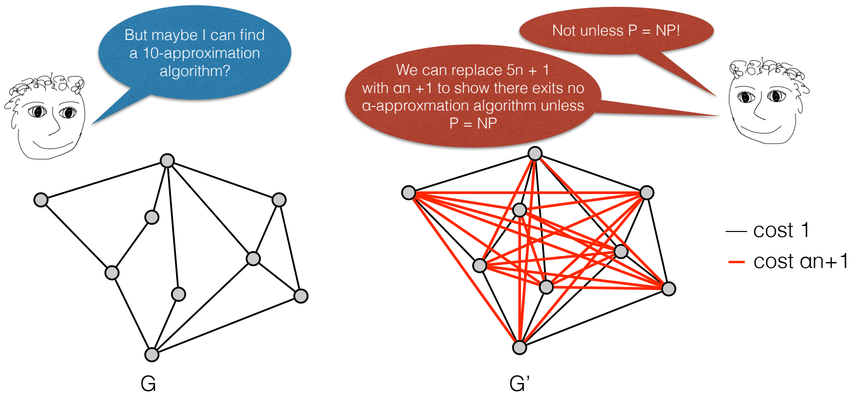


- *G has a Hamiltonian cycle* ⇔ *optimal cost of TSP in G' is n = 9.*
- *G has no Hamiltonian cycle* ⇔ *optimal cost of TSP in G' is at least n -1 + 46*

= 8 + 46 = 54



TSP: Inapproximability



- *G has a Hamiltonian cycle* ⇔ *optimal cost of TSP in G' is n.*
- *G has no Hamiltonian cycle* ⇔ *optimal cost of TSP in G' ≥ n -1 + (αn + 1)*

  $= (\alpha+1)n$

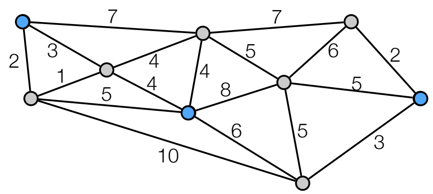
k-center

The k-center problem

- **Input.** An integer k and a complete, undirected graph G=(V,E), with distance d(i,j) between each pair of vertices $i,j \in V$.
- d is a metric:
  - dist(i,i) = 0
  - dist(i,j) = dist(j,i)
  - dist(i,l) ≤ dist(i,j) + dist(j,l)
- **Goal.** Choose a set $S \subseteq V$, $|S| = k$, of k centers so as to minimize the maximum distance of a vertex to its closest center.

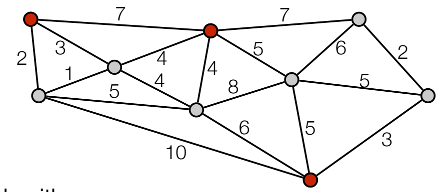
$$S = \text{argmin}_{S \subseteq V, |S|=k} \max_{i \in V} \text{dist}(i,S)$$

- **Covering radius.** Maximum distance of a vertex to its closest center.



k-center: Greedy algorithm

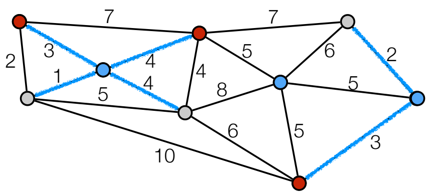
- **Greedy algorithm.**
  - Pick arbitrary i in V.
  - Set S = {i}
  - while |S| < k do
    - Find vertex j farthest away from any cluster center in S
    - Add j to S



- Greedy is a 2-approximation algorithm:
  - polynomial time ✓
  - valid solution ✓
  - factor 2

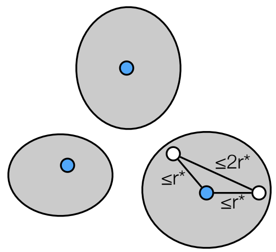
k-center analysis: optimal clusters

- Optimal clusters: each vertex assigned to its closest optimal center.



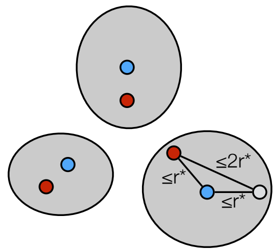
k-center analysis

- r^* optimal radius.
- **Claim:** Two vertices in same optimal cluster has distance at most $2r^*$ to each other.

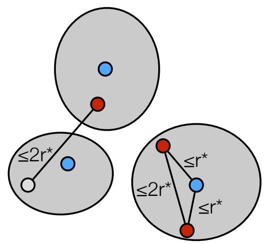


k-center: analysis greedy algorithm

- r^* optimal radius.
- Show all vertices within distance $2r^*$ from a center.
- Consider optimal clusters. 2 cases.
  1. *Algorithm picked one center in each optimal cluster*
     - distance from any vertex to its closest center $\leq 2r^*$.

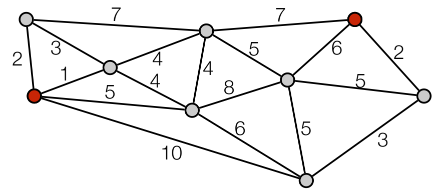


  2. *Some optimal cluster does not have a center.*
     - Some cluster have more than one center.
     - Distance between these two centers $\leq 2r^*$.
     - When second center in same cluster picked it was the vertex farthest away from any center.
     - Distance from any vertex to its closest center at most $2r^*$.



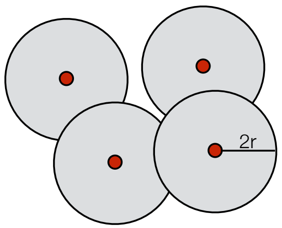
Bottleneck algorithm

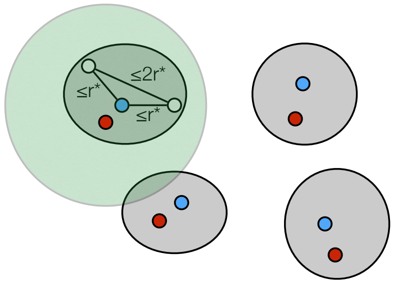
- Assume we know the optimum covering radius r.
- Bottleneck algorithm.
  - Set R := V and S := Ø.
  - while R ≠ Ø do
    - Pick arbitrary i in R.
    - Add j to S
    - Remove all vertices with $d(j,v) \leq 2r$ from R.
- Example: k= 3. r = 4.



Analysis bottleneck algorithm

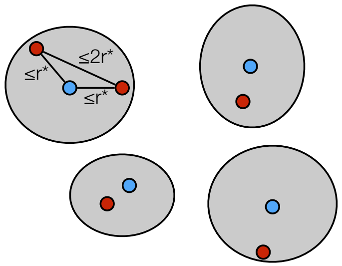
- r* optimal radius.
- Covering radius is at most 2r = 2r*.
- Show that we cannot pick more than k centers:
  - We can pick at most one in each optimal cluster:
    - Distance between two nodes in same optimal cluster ≤ 2r.*
    - When we pick a center in a optimal cluster all nodes in same optimal cluster is removed.





Analysis bottleneck algorithm

- r* optimal radius.
- Can use algorithm to “guess” r* (at most n^2 values).
- If algorithm picked more than k centers then r* > r.
  - If algorithm picked more than k centers then it picked more than one in some optimal cluster.
  - Distance between two nodes in same optimal cluster ≤ 2r.*
  - If more than one in some optimal cluster then 2r < 2r*.



k-center: Inapproximability

- There is no α-approximation algorithm for the k-center problem for α < 2 unless P=NP.
- **Proof.** Reduction from dominating set.
- *Dominating set.* Given G=(V,E) and k. Is there a (dominating) set S ⊆ V of size k, such that each vertex is either in S or adjacent to a vertex in S?
- Given instance of the dominating set problem construct instance of k-center problem:
  - Complete graph G’ on V.
  - All edges from E has weight 1, all new edges have weight 2.
  - Radius in k-center instance 1 or 2.
  - G has an dominating set of size k <=> opt solution to the k-center problem has radius 1.
  - Use α-approximation algorithm A:
    - opt = 1 => A returns solution with radius at most α < 2.
    - opt = 2 => A returns solution with radius at least 2.
    - Can use A to distinguish between the 2 cases.