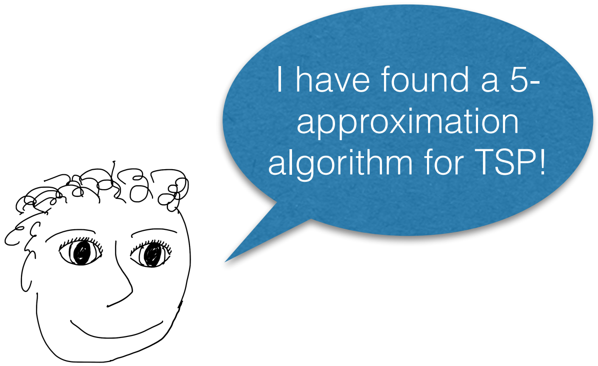
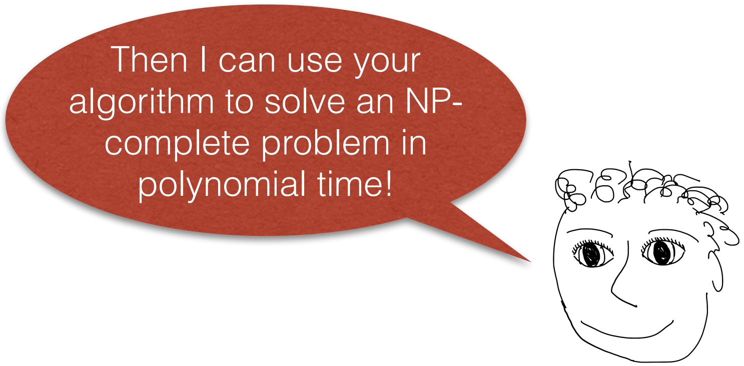
Hardness of Approximation

Inge Li Gørtz

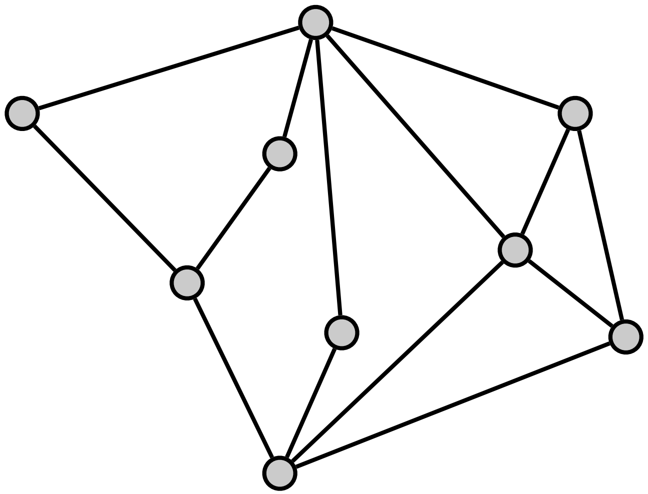
TSP: Inapproximability

- There is no α-approximation algorithm for the TSP for unless P=NP.

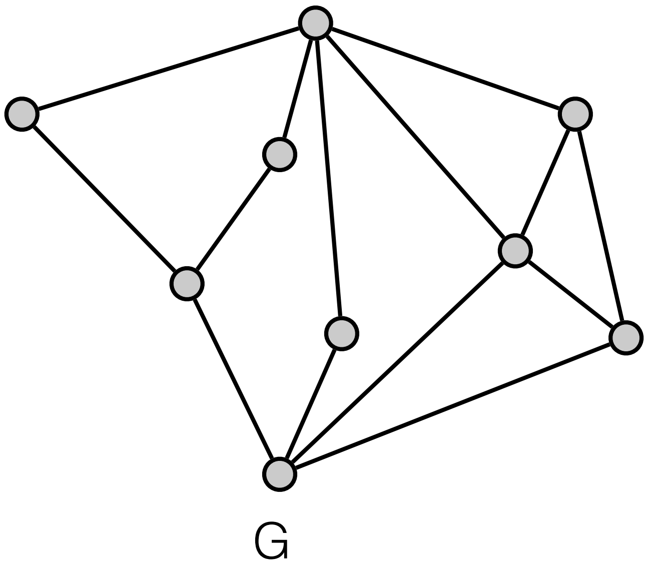


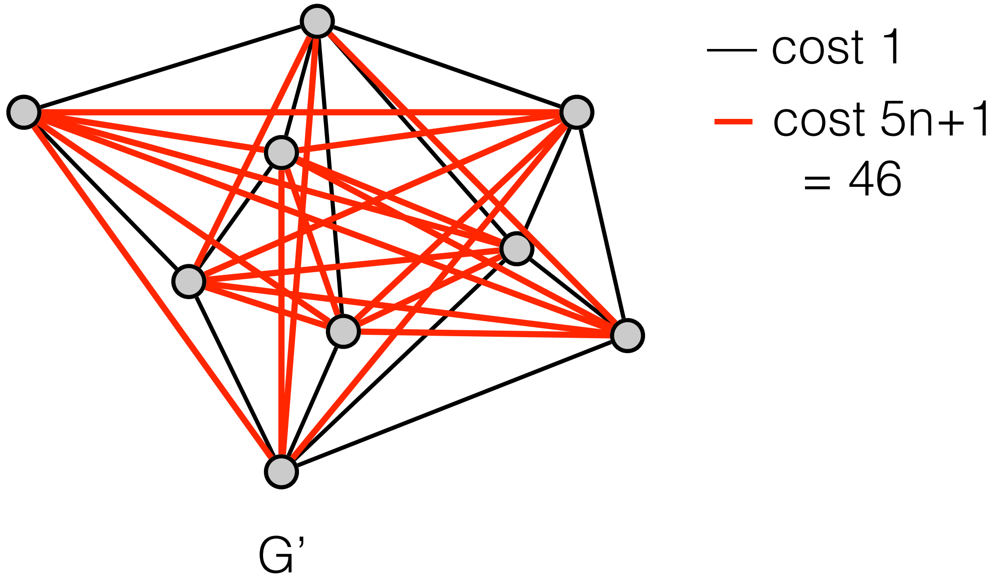


- *Hamiltonian cycle.* Given G=(V,E). Is there a cycle visiting every vertex exactly once?



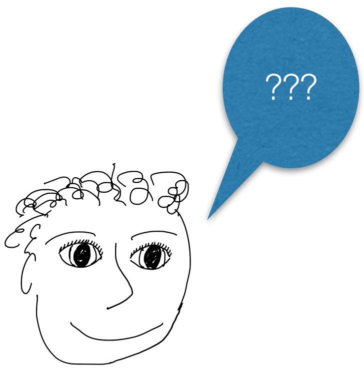
TSP: Inapproximability

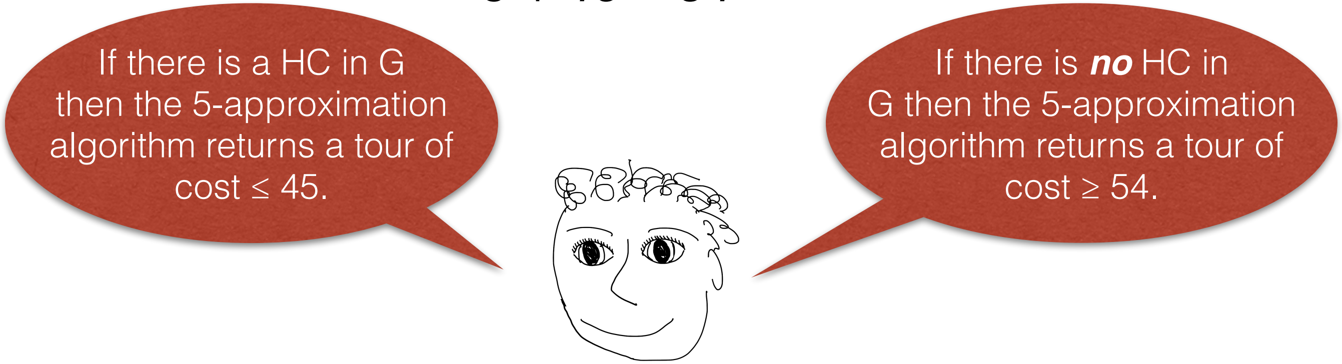




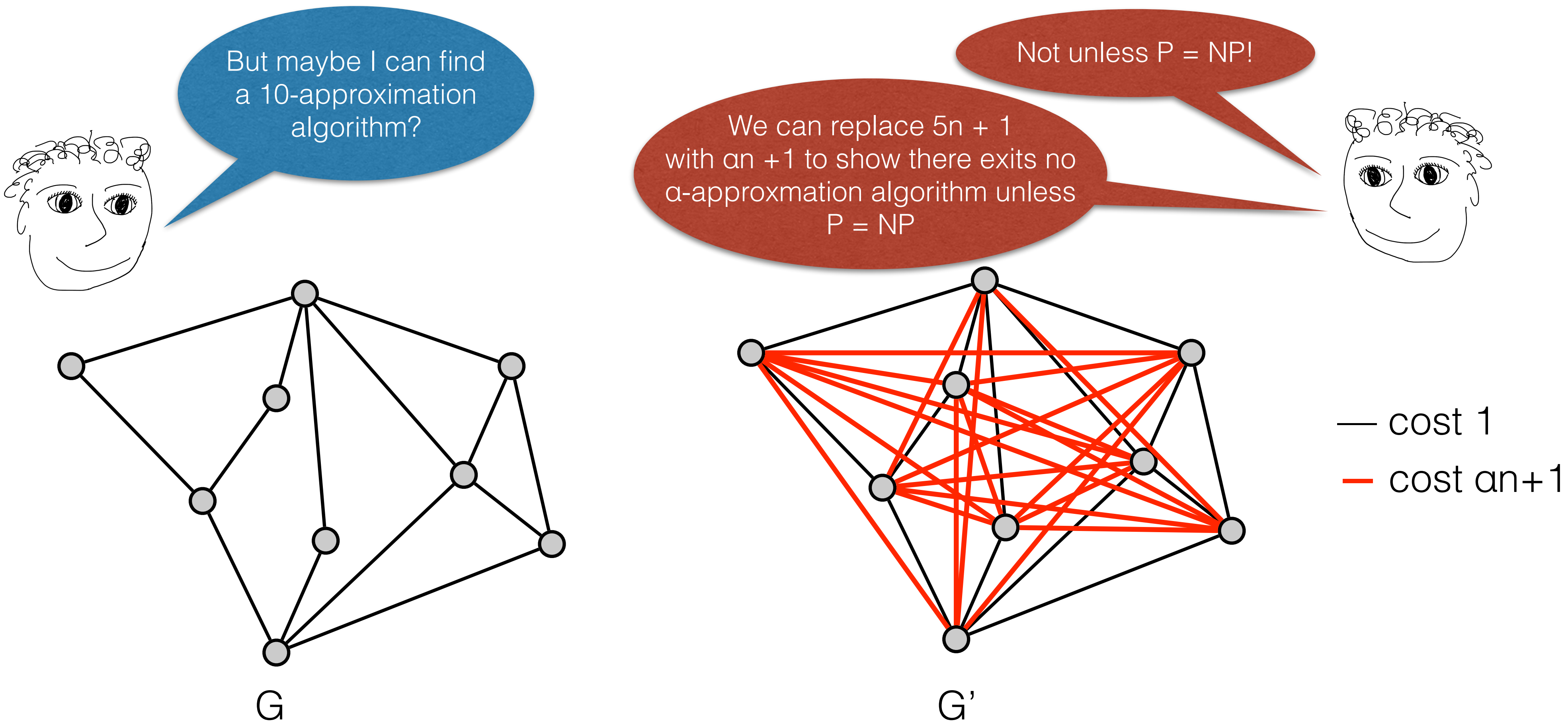
- *G has a Hamiltonian cycle* ⇔ *optimal cost of TSP in G' is n = 9.*
- *G has no Hamiltonian cycle* ⇔ *optimal cost of TSP in G' is at least n -1 + 46*

= *8 + 46 = 54*





TSP: Inapproximability

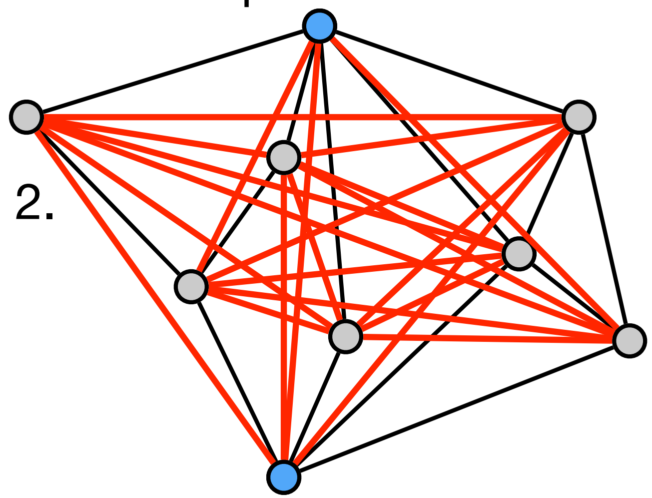


- *G has a Hamiltonian cycle* ⇔ *optimal cost of TSP in G' is n.*
- *G has no Hamiltonian cycle* ⇔ *optimal cost of TSP in G'* $\geq n - 1 + (\alpha n + 1)$

$= (\alpha+1)n$

k-center: Inapproximability

- There is no α-approximation algorithm for the k-center problem for $\alpha < 2$ unless P=NP.
- **Proof.** Reduction from dominating set.
- *Dominating set.* Given G=(V,E) and k. Is there a (dominating) set $S \subseteq V$ of size k, such that each vertex is either in S or adjacent to a vertex in S?
- Given instance of the dominating set problem construct instance of k-center problem:
  - Complete graph G' on V.
  - All edges from E has weight 1, all new edges have weight 2.
  - Radius in k-center instance 1 or 2.
  - G has an dominating set of size k <=> opt solution to the k-center problem has radius 1.
  - Use α-approximation algorithm A:
    - opt = 1 => A returns solution with radius at most $\alpha < 2$.
    - opt = 2 => A returns solution with radius at least 2.
    - Can use A to distinguish between the 2 cases.



Pricing Method: Vertex Cover

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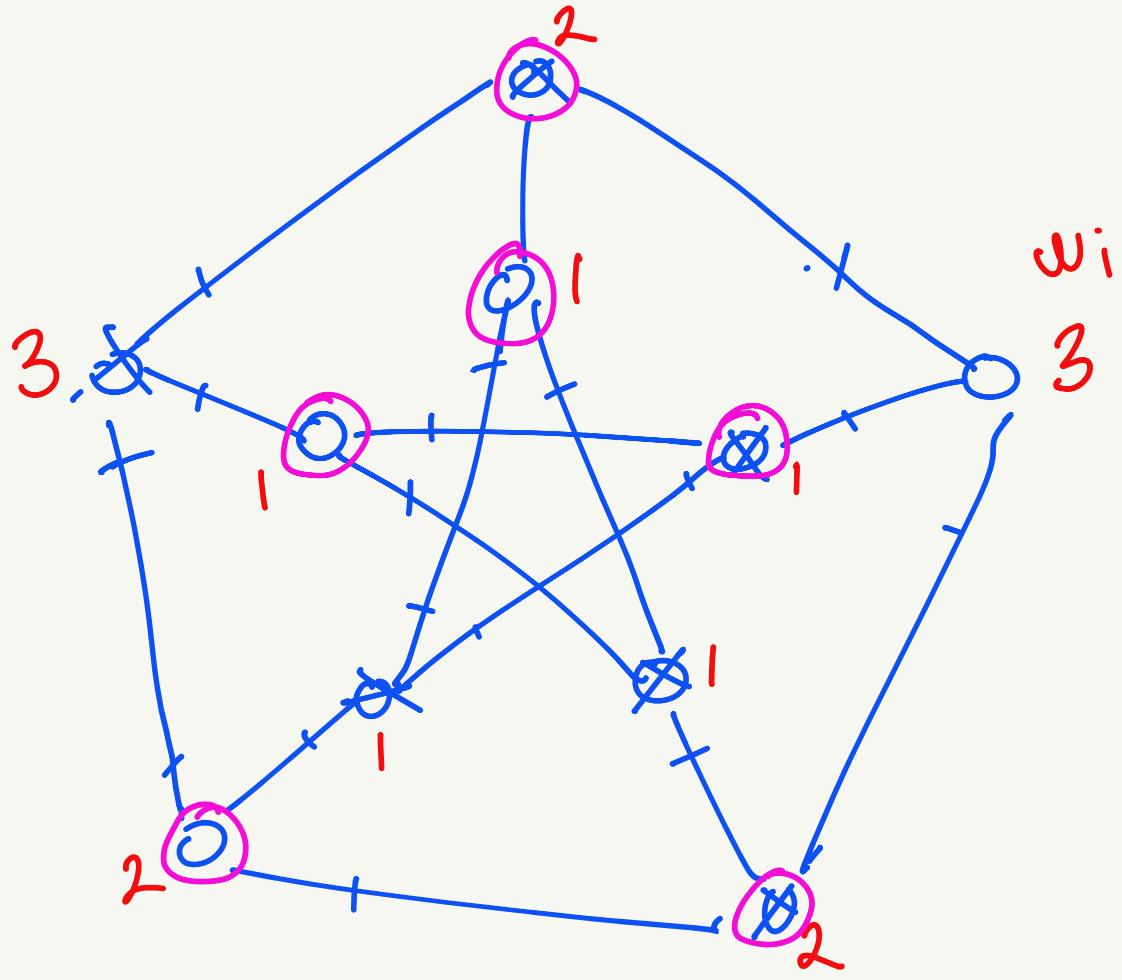
Weighted Vertex Cover



minimize $w(S) = \sum_{i \in S} w_i$

Set Cover?

$G \to I$: $\begin{cases} \text{edges} \simeq \text{elements} \\ \text{vertices} = \text{sets} \end{cases}$

Approximation preserving reduction

since $OPT_{vc}(G) = OPT_{sc}(I)$.

$\Rightarrow$ H_d - approximation

Not approximation preserving: Independent set $\leq_p$ vertex cover

Independent set:

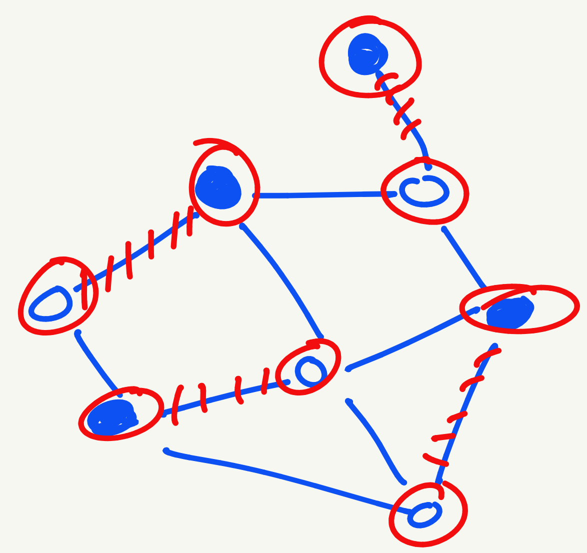
find max ind. set

(set of non-adjacent vertices).

Graph G: I independent

$\Updownarrow$

V - I is a vertex cover



$OPT_{vc} = 4$ $OPT_{IS} = 4$

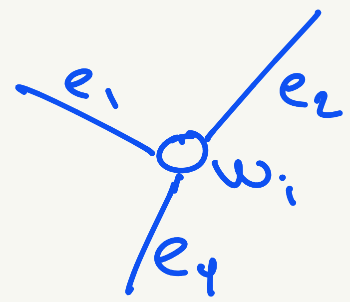
2-approx. VC: 8

IS: 0

pricing method

Each edge pays a price p_e

Fair prices:



$$\sum_{e=(i,j)} p_e \le w_i$$

$$p_1 + p_e + p_4 \le w_i$$

Node i tight (or paid for): $\sum_{e=(i,j)} p_e = w_i$

pricing algorithm

Initially all $p_e = 0$, $S = \emptyset$

while there exists an uncovered edge e:

- raise p_e until some node goes tight.
- add all tight nodes to S

2-approximation: valid ✓, pol. time ✓

all nodes tight but an edge can be paying for two (using its money twice)

$$w(S) = \sum_{i \in S} w_i = \sum_{i \in S} \sum_{e=(i,j)} p_e \le 2 \sum_{e \in E} p_e \le 2\, w(S^*)$$

every edge is in at most 2 sets $\Rightarrow$ edge e contributes at most twice on LHS

where S^* is the optimal solution

Fair prices a lower bound on OPT:

Claim: For fair prices $\sum_{e \in E} p_e \le w(S^*)$ for any vertex cover S^*.

proof:

all edges covered $\Rightarrow$ any edge e covered in at least one set

$$\sum_{e \in E} p_e \le \sum_{i \in S^*} \sum_{e=(i,j)} p_e \le \sum_{i \in S^*} w_i = w(S)$$