# Range Minimum Queries and Lowest Common Ancestor

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## Range Minimum Queries and Lowest Common Ancestor

- Range Minimum Queries (RMQ) and Lowest Common Ancestor (LCA)
- RMQ
  - Simple solutions
  - Better solution
  - 2-level solution
- Reduction between RMQ and LCA

## Range Minimum Queries

- Range minimum query problem. Preprocess array A[1…n] of integers to support
  - RMQ(i,j): return the (entry of) minimum element in A[i…j].

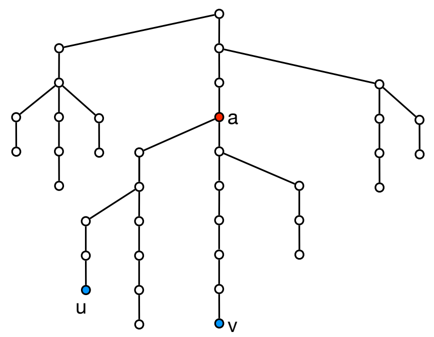
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 7 | 12 | 8 | 2 | 5 | 1 | 4 | 8 | 3 |

  - RMQ(3,6) = 2 (index 5)

- Basic (extreme) solutions
  - Linear search:
    - Space: O(n). Only keep array (no extra space)
    - Time: O(j-i) = O(n)
  - Save all possible answers: Precompute and save all answers in a table.
    - Space: $O(n^2)$ pairs => $O(n^2)$ space
    - Time: O(1)

## Lowest Common Ancestor

- Lowest common ancestor problem. Preprocess rooted tree T with n nodes to support
  - LCA(u,v): return the lowest common ancestor of u and v.




LCA(u,v) = a

## Lowest Common Ancestor

- Basic (extreme) solutions
  - Linear search: Follow paths to root and mark when you visit a node.
    - Space: O(n). Only keep tree (no extra space)
    - Time: O(depth of tree) = O(n)
  - Save all possible answers: Precompute and save all answers in a table.
    - Space: $O(n^2)$ pairs => $O(n^2)$ space
    - Time: O(1)

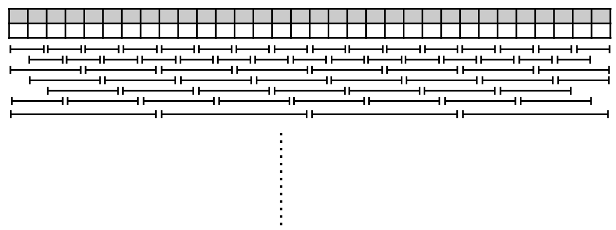
## RMQ and LCA

- Outline.
  - Can solve both RMQ and LCA in linear space and constant time.
    - First solution to RMQ
    - Solution to a special case of RMQ.
    - See that RMQ and LCA are equivalent (can reduce one to the other both ways).

# RMQ

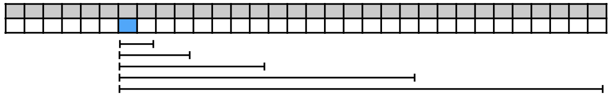
## RMQ: Sparse table solution

- Save the result for all intervals of length a power of 2.



## RMQ: Sparse table solution

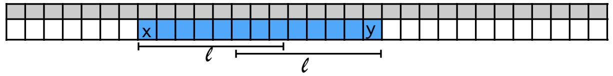
- For all positions we have all power of 2 length intervals starting at that position.



- Space: O(n log n)

## RMQ: Sparse table solution

- Query:




  - Any interval the union of two power of 2 intervals.
  - Query the two intervals and take minimum
- Time: O(1)

## RMQ: Linear space

- Consider ±1RMQ: consecutive entries differ by at most 1.

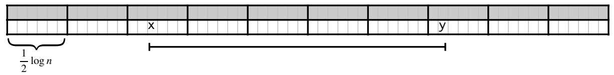
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 5 | 6 | 5 | 4 | 3 | 2 | 3 | 2 | 3 | 4 | 5 | 4 |

- 2-level solution: Combine
  - O(n log n) space, O(1) time
  - O(n²) space, O(1) time.

    ⇓
  - O(n) space, O(1) time.

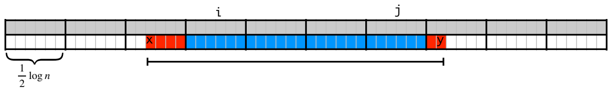
## ±1RMQ

- Divide A into blocks of size $\frac{1}{2}\log n$



## ±1RMQ

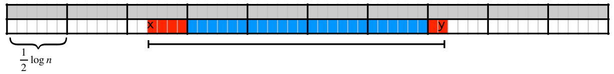
- Divide A into blocks of size $\frac{1}{2}\log n$



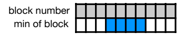

- 2-level data structure:
  - Sparse table on blocks
  - Tabulation inside blocks.

- RMQ(x,y) = min{ RMQ on blocks i to j,
  RMQ inside block i-1,
  RMQ inside block j+1 }.

## ±1RMQ: Data structure on blocks

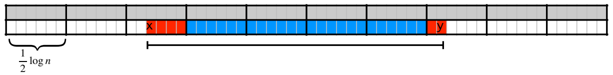



- Two new arrays.
  - Array A': minimum from each block

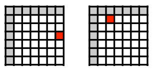



  - B: position in A where A'[i] occurs.
- Sparse table data structure on A'.
- Space: O(|A'| log |A'|) = O(n).
- Time: O(1)

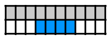
## ±1RMQ: Data structure inside blocks




- Precompute and save all answers for each block.



- Gives solution using
  - Space: O(n) + space for precomputed tables.



  - Time: O(1) + O(1) + O(1) = O(1).
    - O(1): 2 table lookups
    - O(1): sparse table
    - O(1): min{•,•,•}

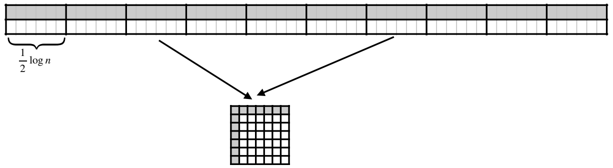
## ±1RMQ: Storing the tables

- Naively: $\log^2 n$ for each table => n log n space. 🙁

- Observation: If X[i] = Y[i] + c then all RMQ answers are the same for X and Y.
  - X = [7, 6, 5, 6, 5, 4]
  - Y = [3, 2, 1, 2, 1, 0]
- Normalize blocks:
  - X = [0, -1, -2, -1, -2, -3] = Y
- Normalized block described by sequence of +1s and -1s:
  - X = Y = -1, -1, +1, -1, -1.
- How many different normalized blocks are there?
  - length of sequence = $\frac{1}{2}\log n - 1$
  - #sequences = $2^{\frac{1}{2}\log n - 1} \le \sqrt{n}$.

## ±1RMQ: Data structure inside blocks

- Precompute and save all answers for each normalized block.
- Size of a table: $O(\log^2 n)$
- For each block save which precomputed table it uses.




- Space: $O(\sqrt{n} \cdot \log^2 n) + O(n/\log n) = O(n)$
- Plugging into 2-level solution:
  - Space: O(n) + space for precomputed tables = O(n).

## LCA and RMQ

## RMQ and LCA

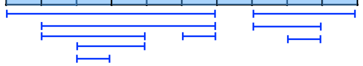
- We will show
  - RMQ —reduces to→ LCA —reduces to→ ±1RMQ

If there is a solution to LCA using s(n) space and t(n) time, then there is a solution to RMQ using O(s(n)) space and O(t(n)) time.

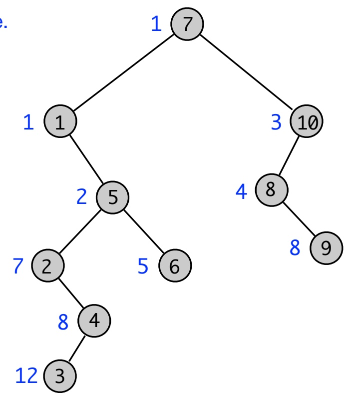
If there is a solution to ±1RMQ using s(n) space and t(n) time, then there is a solution to LCA using O(s(n)) space and O(t(n)) time.

## RMQ to LCA

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 7 | 12 | 8 | 2 | 5 | 1 | 4 | 8 | 3 |



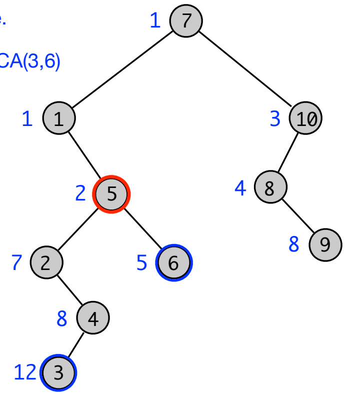
- Cartesian tree.



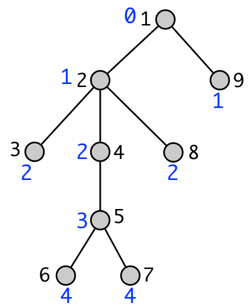
## RMQ to LCA

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 7 | 12 | 8 | 2 | 5 | 1 | 4 | 8 | 3 |

- Cartesian tree.
- RMQ(3,6) = LCA(3,6)




## LCA to ±1RMQ




- E =

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 2 | 4 | 5 | 6 | 5 | 7 | 5 | 4 | 2 | 8 | 2 | 1 | 9 | 1 |

- A =

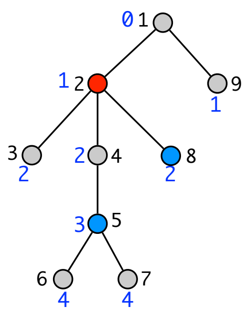
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 1 | 2 | 3 | 4 | 3 | 4 | 3 | 2 | 1 | 2 | 1 | 0 | 1 | 0 |

- R =

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 5 | 6 | 7 | 9 | 13 | 16 |

- E: Euler tour representation: preorder walk, write node preorder number of node when met.
- A: depth of node node in E[i].
- R: first occurrence in E of node with preorder number i
- LCA(i, j) = E[$RMQ_A$(R[i], R[j])].

## LCA to ±1RMQ




- LCA(5,8) = $RMQ_A$(6, 13).

- E =

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 2 | 4 | 5 | 6 | 5 | 7 | 5 | 4 | 2 | 8 | 2 | 1 | 9 | 1 |

- A =

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 1 | 2 | 3 | 4 | 3 | 4 | 3 | 2 | 1 | 2 | 1 | 0 | 1 | 0 |

- R =

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 5 | 6 | 7 | 9 | 13 | 16 |

- E: Euler tour representation: preorder walk, write node preorder number of node when met.
- A: depth of node node in E[i].
- R: first occurrence in E of node with preorder number i
- LCA(i, j) = E[$RMQ_A$(R[i], R[j])].

## RMQ and LCA

- Theorem. RMQ and LCA can be solved in O(n) space and O(1) query time.