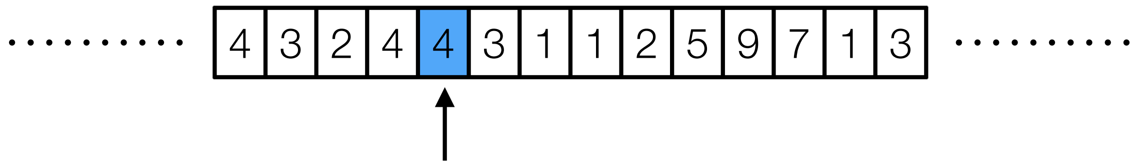
Streaming

Inge Li Gørtz

Streaming model (one-pass)

- Stream.
  - Elements $a_1, a_2, \ldots, a_m$ from the universe $[n] = \{1,2,\ldots,n\}$.
  - Elements arrive one by one.
  - Must process element a_i before we see a_{i+1}.



- Space. Measured in bits.
- Goal. Small space (sublinear/polylogarithmic).
- Example. What can we do in $O(\log n + \log m)$ space?

Frequent elements

Frequent elements

- Heavy Hitters Problem. Find all elements i that occurs more than m/k times for some fixed k.

- Example. Return all elements that occur more than 21/3 times = 7.

4	4	1	2	4	4	3	1	1	2	5	9	7	4	1	3	4	1	4	4	1

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- Bad news. Need $\Omega(n)$ space for one-pass algorithm.

- Good news.
  - Can estimate the frequency.
  - Can do better if we allow one-sided error:
    - Output all elements that occur more than m/k times.
    - Might also output other elements.

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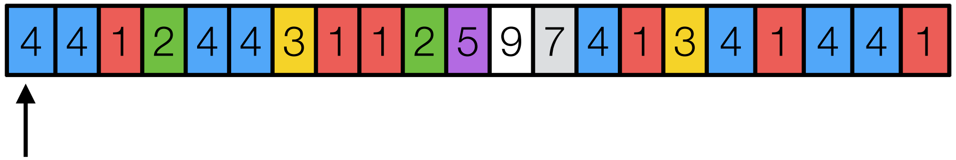
- Misra-Gries.

Keep k-1 counters in an associative array A.
while (stream is not empty) do
    If $j \in \text{keys}(A)$ then
        $A[j] \leftarrow A[j] + 1$
    else if $|\text{keys}(A)| < k - 1$ then
        $A[j] \leftarrow 1$
    else
        Decrement all counters by 1.
        Remove all elements with counter 0.
Output all elements in $\text{keys}(A)$

- Space. $O(k \cdot (\log n + \log m))$.

Frequent elements

- Example. k = 3.



counter 1: 4 , 2

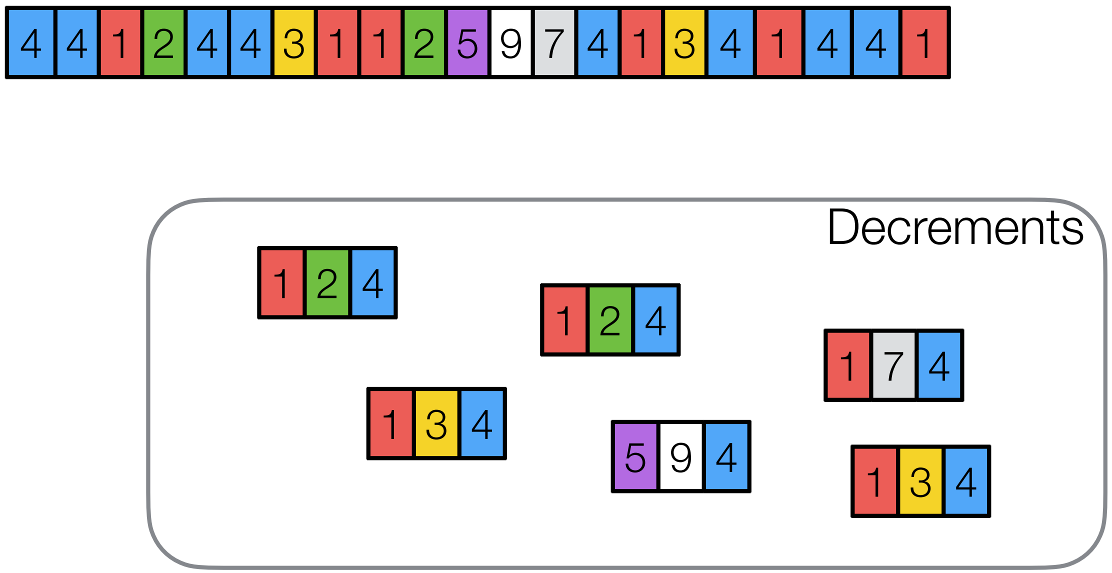
counter 2: 1 , 1

Keep k-1 counters in an associative array A.
while (stream is not empty) **do**
  If $j \in$ keys(A) **then**
    $A[j] \leftarrow A[j] + 1$
  else if $|$keys$(A)| < k - 1$ **then**
    $A[j] \leftarrow 1$
  else
    Decrement all counters by 1.
    Remove all elements with counter 0.
Output all elements in keys(A)

Misra-Gries Analysis

- Lemma. Any item with frequency more than m/k is in A by the end of the algorithm.
- Lemma. Let $\hat{f}_i$ be the estimate of the frequency of element i. Then

$$f_i - \frac{m}{k} \leq \hat{f}_i \leq f_i \;.$$



Reservoir Sampling

Reservoir Sampling

- Algorithm.

> put the first k elements into a "reservoir" $R = \{r_1, r_2, \ldots, r_k\}$.
> for $i > k$ until the stream is empty do
>   with probability k/i replace a random entry of R with a_i
> Return R.

- Claim. For all $t \geq i$, $P[a_i \in R_t] = k/t$, where R_t denotes the reservoir after time t.

- Proof. By induction on t.

    - Base case: $t = k$ (assume stream has length at least k).

$$P[x_i \in R_t] = 1 = k/t\,.$$

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- Claim. For all $t \geq i$, $P[a_i \in R_t] = k/t$, where R_t denotes the reservoir after time t.

- Proof. By induction on t. Assume $t > k$.

  - Case 1. $i = t$.

$$P[a_i \in R_t] = P[a_t \in R_t] = k/t\,.$$

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- Algorithm.

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> for $i > k$ until the stream is empty do
>   with probability k/i replace a random entry of R with a_i
> Return R.

- Claim. For all $t \geq i$, $P[a_i \in R_t] = k/t$, where R_t denotes the reservoir after time t.

- Proof. By induction on t. Assume $t > k$. Case 2. $i < t$.

$$P[x_i \in R_t] = P[x_i \in R_{t-1}] \cdot P[x_i \text{ not replaced at time } t \,|\, x_i \in R_{t-1}]$$

$$= \frac{k}{t-1} \cdot P[x_i \text{ not replaced at time } t \,|\, x_i \in R_{t-1}]$$

$$= \frac{k}{t-1}\left(1 - \frac{k}{t} \cdot \frac{1}{k}\right) = \frac{k}{t}$$