Streaming: Sketching

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Today

- Sketching
- CountMin sketch

Sketching

Sketching

- **Sketching.** create compact sketch/summary of data.
- **Example.** Durand and Flajolet 2003.
  - Condensed the whole Shakespeares' work

```
ghfffghfghgghggggghghheehfhfhhgghghghhfgffffhhhiigfhhffgfiihfhhh
igigighfgihfffghigihghigfhhgeegeghgghhhgghhfhidiigihighihehhhfgg
hfgighigffghdieghhhggghhfghhfiiheffghghihifgggffihgihfggighgiiif
fjgfgjhhjiifhjgehgghfhhfhjhigggghghihigghhihihgiighgfhlgjfgjjjmfl
```

  - Estimated number of distinct words: 30897 (correct answer is 28239, ie. relative error of 9.4%).
- **Composable.**
  - Data streams S_1 and S_2 with sketches $sk(S_1)$ and $sk(S_2)$
  - There exists an efficiently computable function f such that

$$sk(S_1 \cup S_2) = f(sk(S_1), sk(S_2))$$

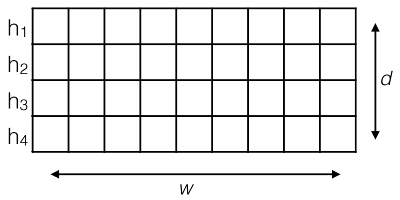
CountMin Sketch

Frequency Estimation

- Frequency estimation. Construct a sketch such that can estimate the frequency f_i of any element $i \in [n]$.

CountMin Sketch

- Fixed array of counters of width w and depth d. Counters all initialized to be zero.
- Pariwise independent hash function for each row $h_i : [n] \rightarrow [w]$.
- When item x arrives increment counter $h_i(x)$ of in all rows.



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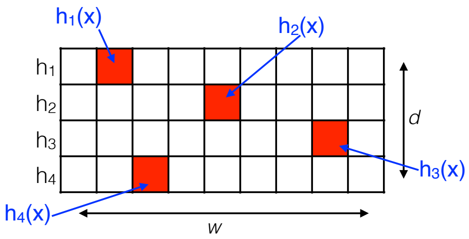

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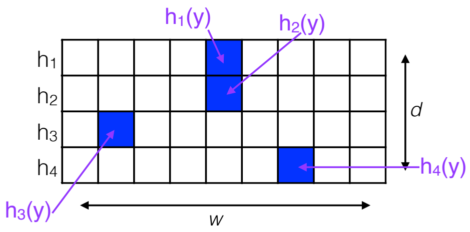
- Estimate frequency of y: return minimum of all entries y hash to.



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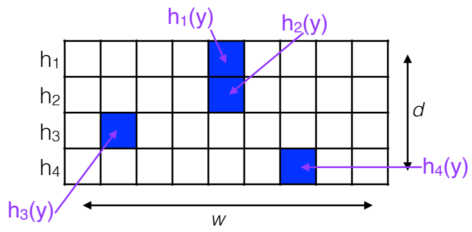
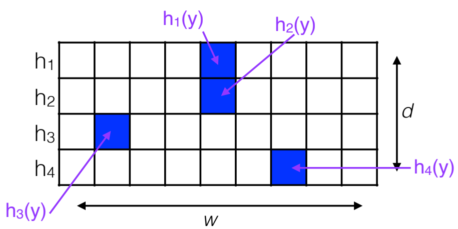


CountMin Sketch

Algorithm 1: CountMin

Initialize d independent hash functions $h_j : [n] \to [w]$.
Set counter $C_j(b) = 0$ for all $j \in [d]$ and $b \in [w]$.
while *Stream S not empty* **do**
  if *Insert(x)* **then**
    for $j = 1 \ldots n$ **do**
      $C_j(h_j(x)) = +1$
    end
  else if *Frequency(i)* **then**
    return $\hat{f}_i = \min_{j \in [d]} C_j(h_j(i))$.
  end
end

- The estimator $\hat{f}_i$ has the following property:
  - $\hat{f}_i \geq f_i$
  - $\hat{f}_i \leq f_i + 2m/w$ with probability at least $1 - (1/2)^d$



CountMin Sketch: Analysis

- **Claim.** $\hat{f}_i \leq f_i + 2m/w$ **with probability at least** $1 - (1/2)^d$
- Consider a fixed element i. $Z_j = C(h_j(i))$
- Let $b = h_j(i)$. Then

$$Z_j = \sum_{s:h_j(s)=b} f_s$$

- Expected value of Z_j

$$E[Z_j] = E\left[\sum_{s:h_j(s)=b} f_s\right] \qquad = f_i + \frac{1}{w}\sum_{s:s\neq i} f_s \qquad \leq f_i + \frac{m}{w}$$

- Want to bound

$$P[Z_j \geq f_i + 2m/w] = P[Z_j - f_i \geq 2m/w]$$

$$= \leq \frac{E[Z_j - f_i]}{2m/w} = \frac{E[Z_j] - f_i}{2m/w} \leq \frac{(f_i + m/w) - f_i}{2m/w} = \frac{1}{2}$$

CountMin Sketch: Analysis

- **Claim.** $\hat{f}_i \leq f_i + 2m/w$ **with probability at least** $1 - (1/2)^d$
- Consider a fixed element i. We have $P[Z_j - f_i \geq 2m/w] \leq 1/2$.
- What is the probability that $\hat{f}_i \geq f_i + 2m/w$?
- $P[\hat{f}_i \geq f_i + 2m/w] = P[Z_1 \geq f_i + 2m/w$

$$= P[Z_1 \geq f_i + 2m/w] \cdot P[Z_2 \geq f_i + 2m/w] \cdot \cdots \cdot P[Z_d \geq f_i + 2m/w]$$

$$\leq \frac{1}{2} \cdot \frac{1}{2} \cdot \cdots \cdot \frac{1}{2}$$

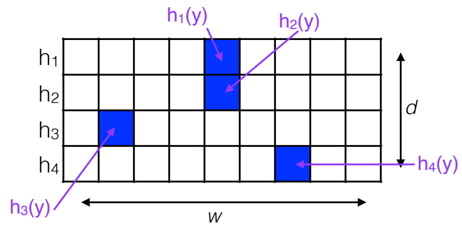
$$= \left(\frac{1}{2}\right)^d$$

- Thus

$$P[\hat{f}_i \leq f_i + 2m/w] \geq 1 - \left(\frac{1}{2}\right)^d$$

CountMin Sketch: Analysis

- Use $w = 2/\varepsilon$ and $d = \lg(1/\delta)$.
- The estimator $\hat{f}_i$ has the following property:
  - $\hat{f}_i \geq f_i$
  - $\hat{f}_i \leq f_i + \varepsilon m$ with probability at least $1 - \delta$
- **Space.** $O(dw) = O(2\lg(1/\delta)/\varepsilon) = O(\lg(1/\delta)/\varepsilon)$ words.
- **Query and processing time.** $O(d) = O(\lg(1/\delta))$



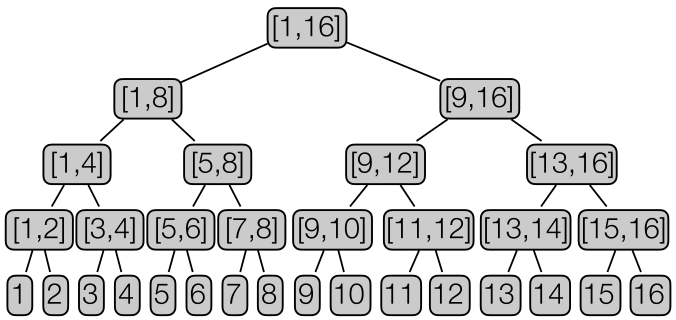
Applications of CountMin Sketch

- We can use the CountMin Sketch to solve e.g.:
  - **Heavy hitters:** List all heavy hitters (elements with frequency at least m/k).
  - **Range(a,b):** Return (an estimate of) the number of elements in the stream with value between a and b.
- Exercise.
  - How can we solve heavy hitters with a single CountMin sketch?
  - What is the space and query time?

Dyadic Intervals

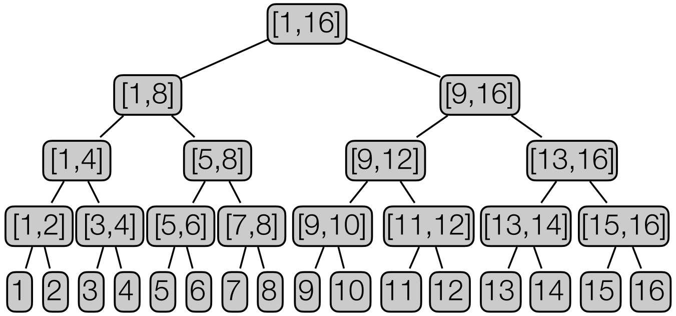
- **Dyadic intervals.** Set of intervals:

$$\{[j\frac{n}{2^i}+1, \ldots, (j+1)\frac{n}{2^i}] \mid 0 \leq i \leq \lg n,\ 0 \leq j \leq 2^{i-1}\}$$



Heavy Hitters

- **Heavy Hitters.** Store a CountMin Sketch for each level in the tree of dyadic intervals
  - On a level: Treat all elements in same interval as the same element.



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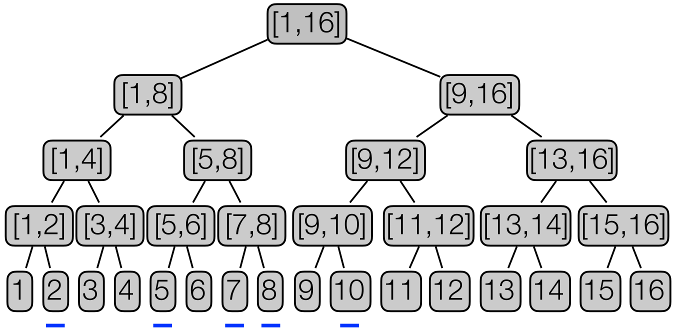

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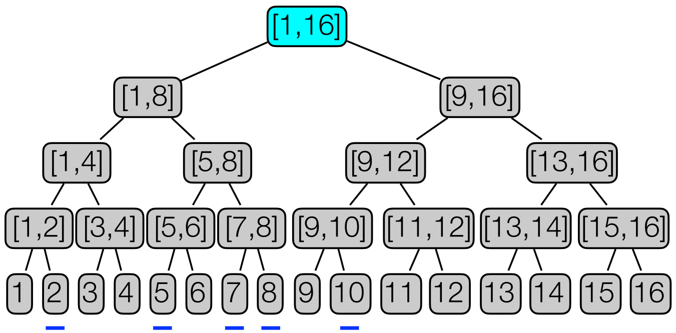
- To find heavy hitters:
  - traverse tree from root.
  - only visit children with frequency ≥ m/k.



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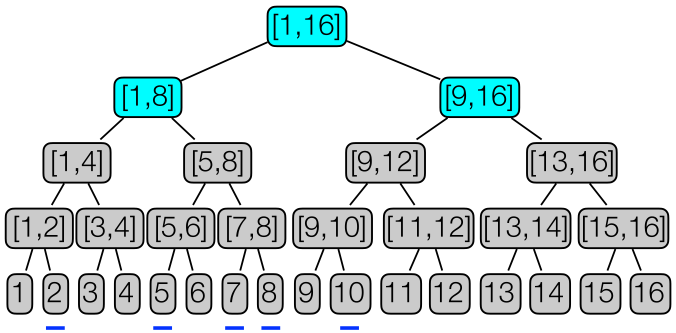

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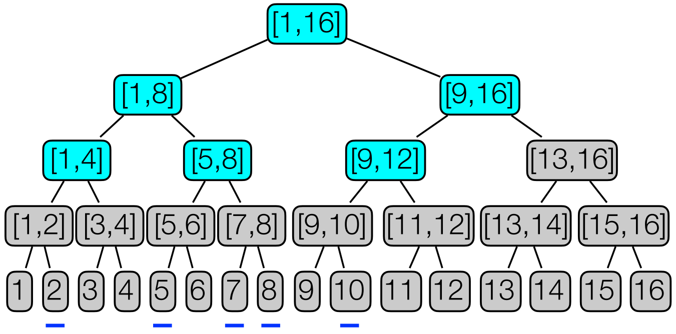

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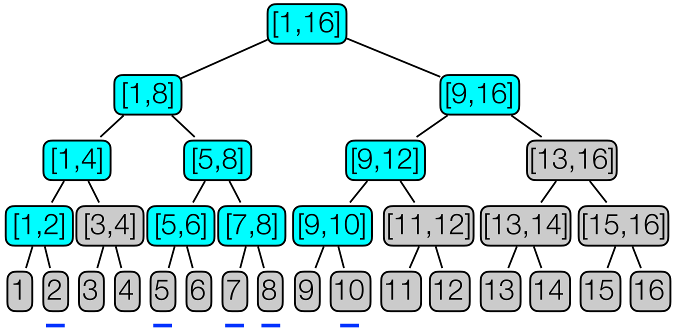

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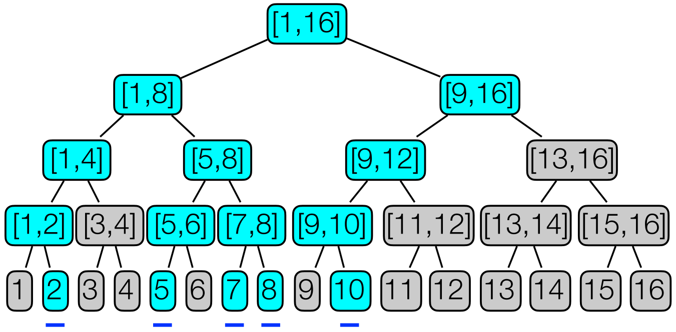

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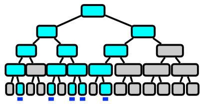

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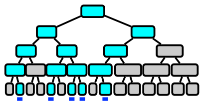
- **Analysis.**
  - **Time.**
    - Number of intervals queried: $O(k \lg n)$.
    - Query time: $O(k \lg n \cdot \lg(1/\delta))$
  - **Space.**

$$O\left(\lg n \cdot \frac{1}{\epsilon} \lg\left(\frac{1}{\delta}\right)\right)$$ words.



Count Sketch

Algorithm 2: CountSketch

```
Initialize d independent hash functions h_j : [n] → [w].
Initialize d independent hash functions s_j : [n] → {±1}.
Set counter C[j, b] = 0 for all j ∈ [d] and b ∈ [w].
while Stream S not empty do
    if Insert(x) then
        for j = 1 ... d do
            C[j, h_j(x)] =+ s_j(i)
        end
    else if Frequency(i) then
        f̂_ij = C(h_j(i)) · s_j(i)
        return f̃_ij = median_{j∈[d]} f̂_ij
    end
end
```

	Space	Error
Count-Min	$O\left(\frac{1}{\epsilon}\log n\right)$	ϵF_1 (one-sided)
Count-Sketch	$O\left(\frac{1}{\epsilon^2}\log n\right)$	$\epsilon\sqrt{F_2}$ (two-sided)