Streaming

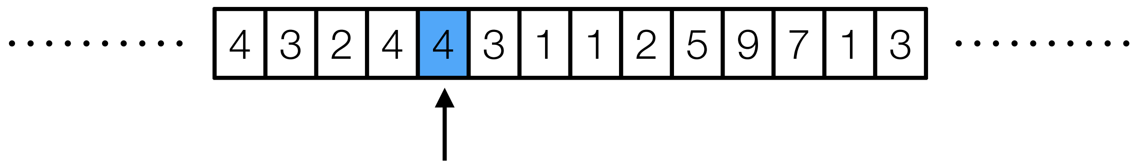
Inge Li Gørtz

Today

- Streaming model
- Frequent Elements (Misra-Gries)
- Reservoir Sampling

Streaming model (one-pass)

- Stream.
    - Elements $a_1, a_2, \ldots, a_m$ from the universe $[n] = \{1,2,\ldots,n\}$.
    - Elements arrive one by one.
    - Must process element a_i before we see a_{i+1}.



- Space. Measured in bits.
- Goal. Small space (sublinear/polylogarithmic).
- Example. What can we do in $O(\log n + \log m)$ space?

Frequent elements

Frequent elements

- Heavy Hitters Problem. Find all elements *i* that occurs more than *m/k* times for some fixed *k*.

- Example. Return all elements that occur more than 21/3 times = 7.

4	4	1	2	4	4	3	1	1	2	5	9	7	4	1	3	4	1	4	4	1

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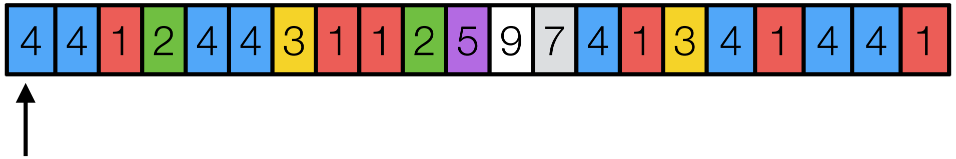
- Bad news. Need $\Omega(n)$ space for one-pass algorithm.

- Good news.
  - Can estimate the frequency.
  - Can do better if we allow one-sided error:
    - Output all elements that occur more than m/k times.
    - Might also output other elements.

Frequent elements

- Example. k = 3.



counter 1: 4 , 2

counter 2: 1 , 1

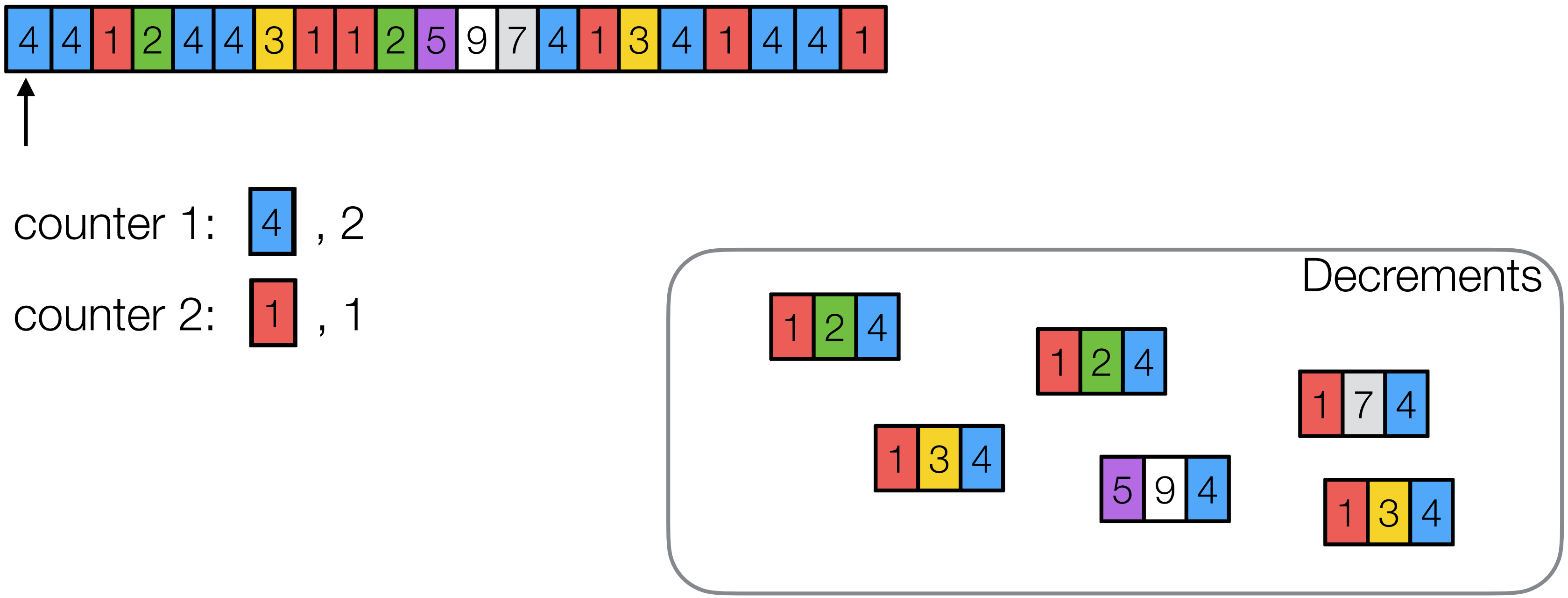
Keep k-1 counters in an associative array A.
while (stream is not empty) do
    If $j \in$ keys(A) then
        $A[j] \leftarrow A[j] + 1$
    else if $|$keys$(A)| < k - 1$ then
        $A[j] \leftarrow 1$
    else
        Decrement all counters by 1.
        Remove all elements with counter 0.
Output all elements in keys(A)

- Space. $O(k \cdot (\log n + \log m))$.

Misra-Gries Analysis

- Lemma. Any item with frequency more than m/k is in A by the end of the algorithm.
- Lemma. Let $\hat{f}_i$ be the estimate of the frequency of element i. Then

$$f_i - \frac{m}{k} \le \hat{f}_i \le f_i \ .$$



Reservoir Sampling

Reservoir Sampling

- Algorithm.

> put the first k elements into a "reservoir" $R = \{r_1, r_2, \ldots, r_k\}$.
> for $i > k$ until the stream is empty do
>   with probability k/i replace a random entry of R with a_i
> Return R.

- Claim. For all $t \geq i$, $P[a_i \in R_t] = k/t$, where R_t denotes the reservoir after time t.

- Proof. Consider element a_i.
  - $P[a_i$ chosen at time $i] = k/i$.
  - $P[a_i$ replaced at time $j] = (k/j) \cdot (1/k) = 1/j$.
  - $P[a_i$ not replaced at time $j] = 1 - 1/j = (j-1)/j$.
  - Thus

$$P[a_i \in R_t] = \frac{k}{i} \cdot \frac{i}{i+1} \cdot \frac{i+1}{i+2} \cdots \frac{t-1}{t} = \frac{k}{t} .$$