Distance Oracles

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Algorithmic Techniques for Modern Data Models
DTU

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Overview for today

- Shortest paths and distance oracles

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- Approximate distance oracles

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  - Bounding stretch

The shortest path problem

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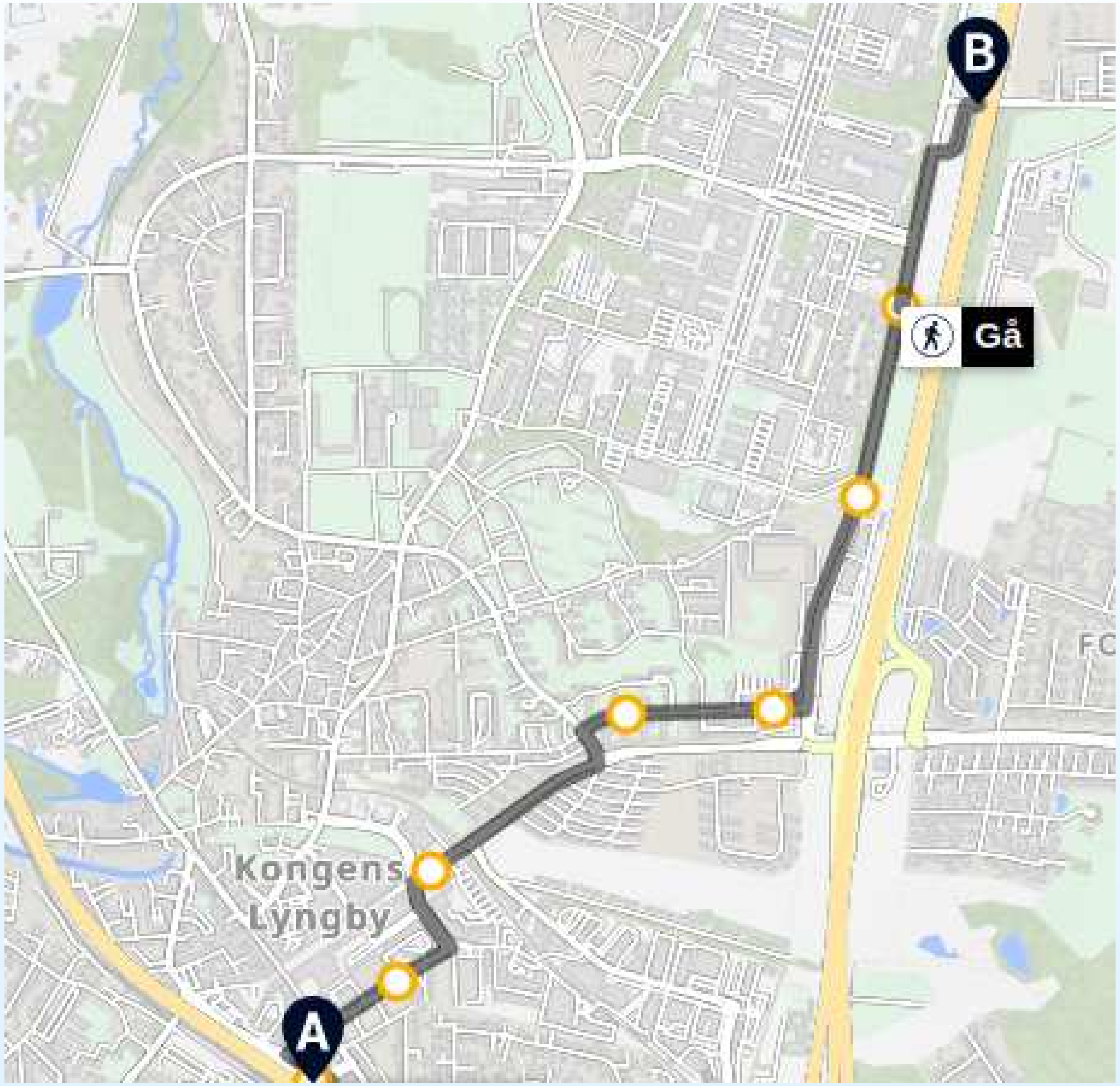

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- We want a data structure for G that can answer queries of the form “What is $\delta(s, t)$?” for any $s, t \in V$

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- We therefore consider *approximate* shortest path distances

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- For any $u, v \in V$, $\tilde{\delta}(u,v)$ is an estimate of $\delta(u,v)$ of *stretch* $t \geq 1$ if

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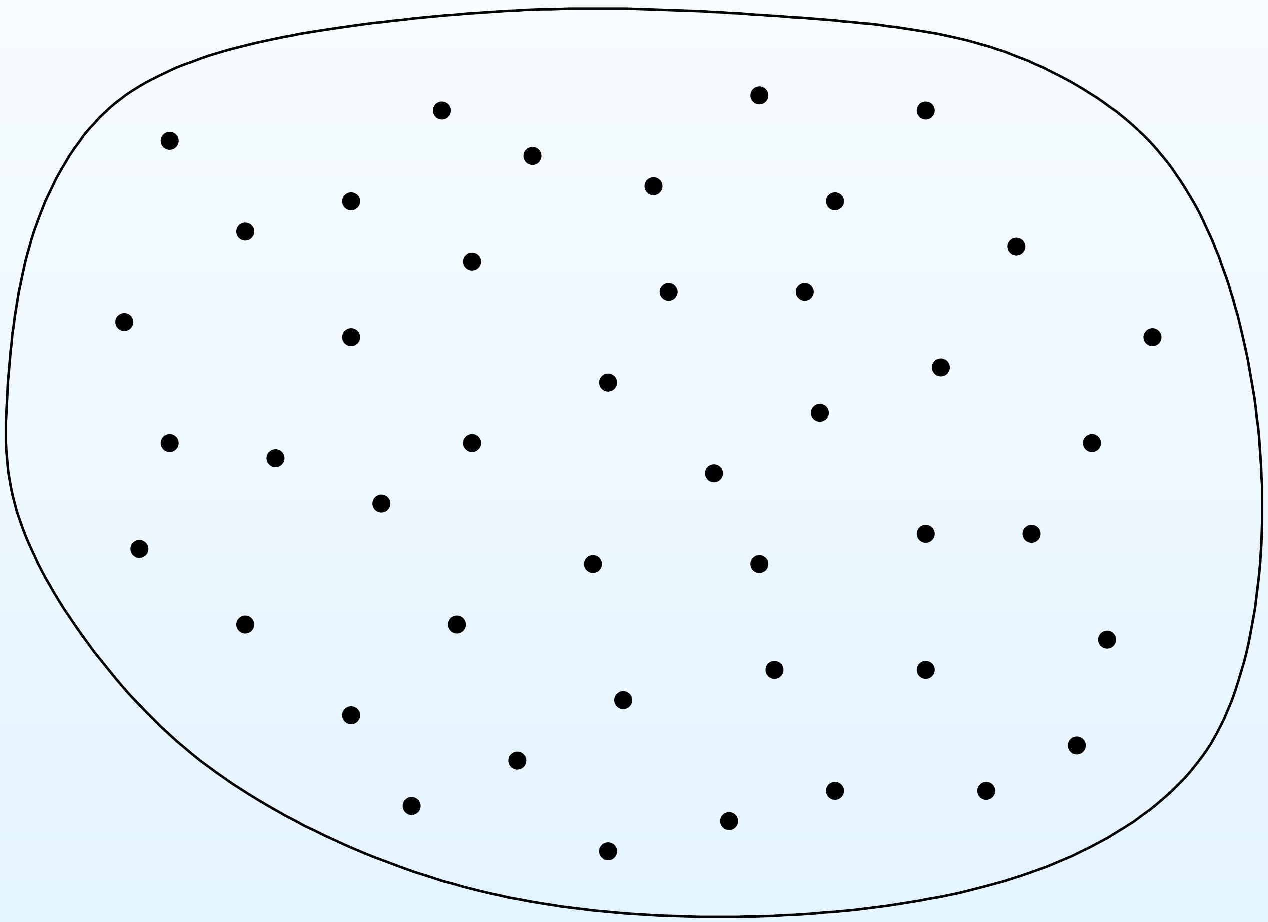
- t-approximate distance oracle:
  - Answers queries with estimates of stretch t
  - Should preferably use a small amount of space
- From now on, we consider only *undirected* edge-weighted graphs since for any stretch t, $\Theta(n^2)$ space is the best we can hope for for directed graphs

A simple approximate distance oracle

- Sample each vertex independently with some probability p:



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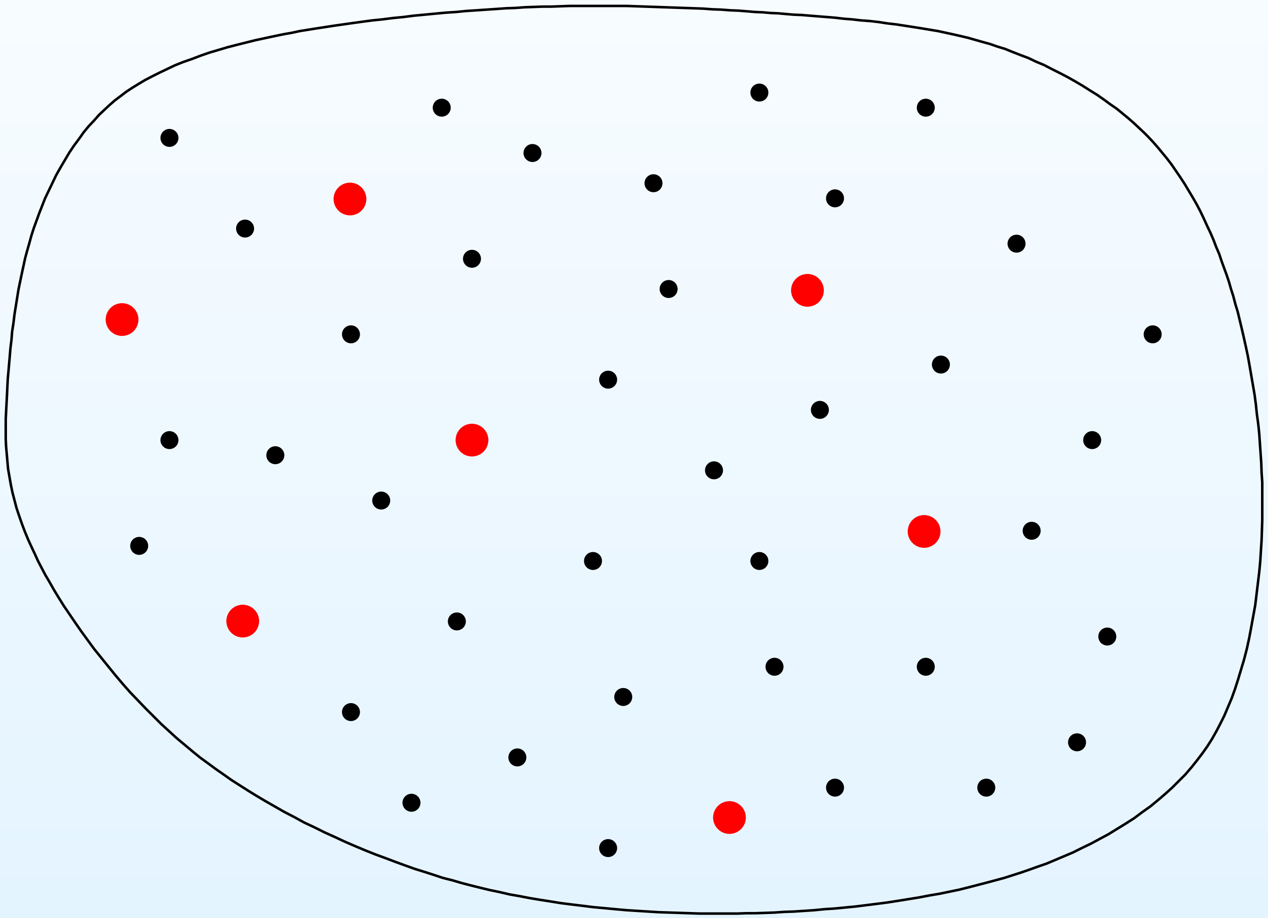

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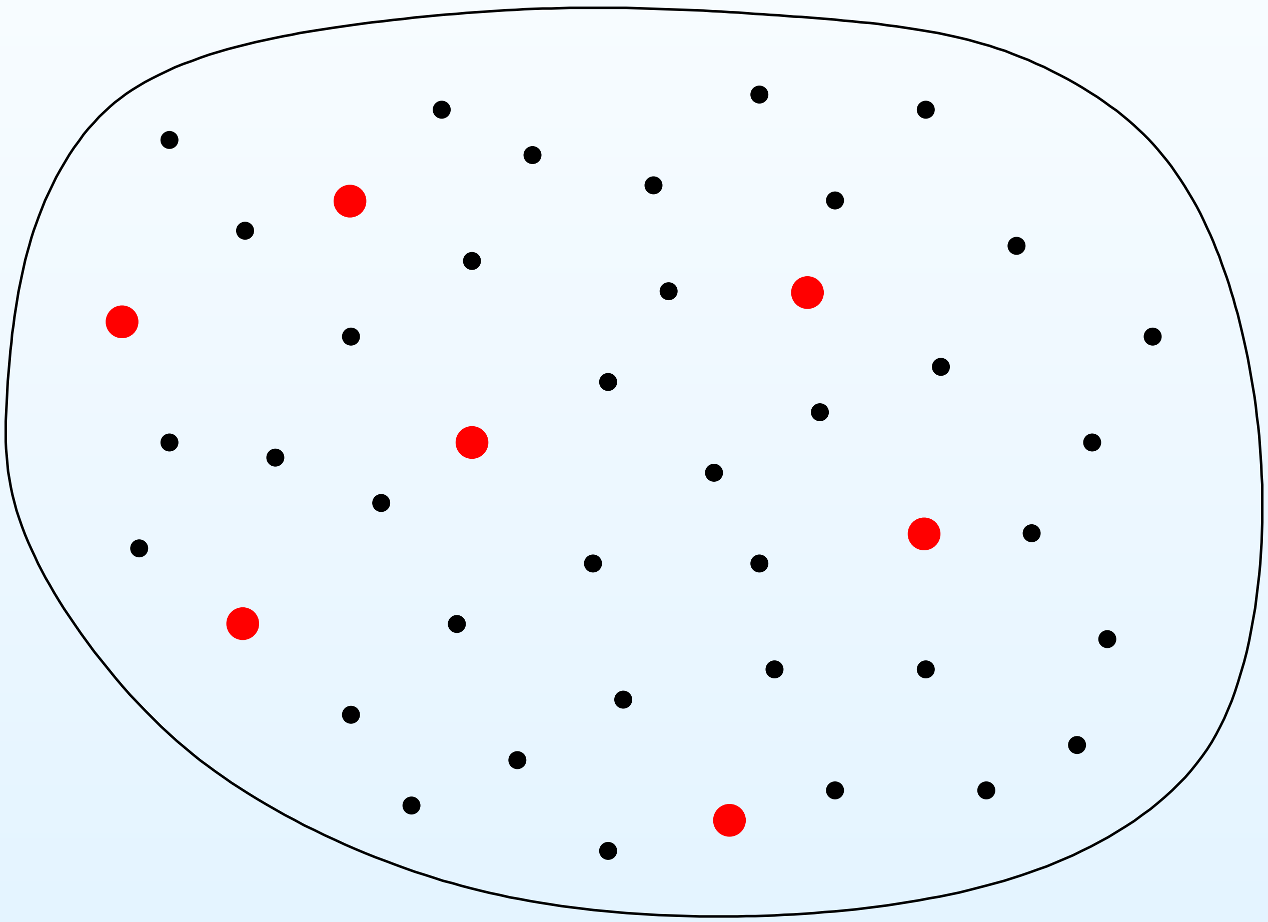

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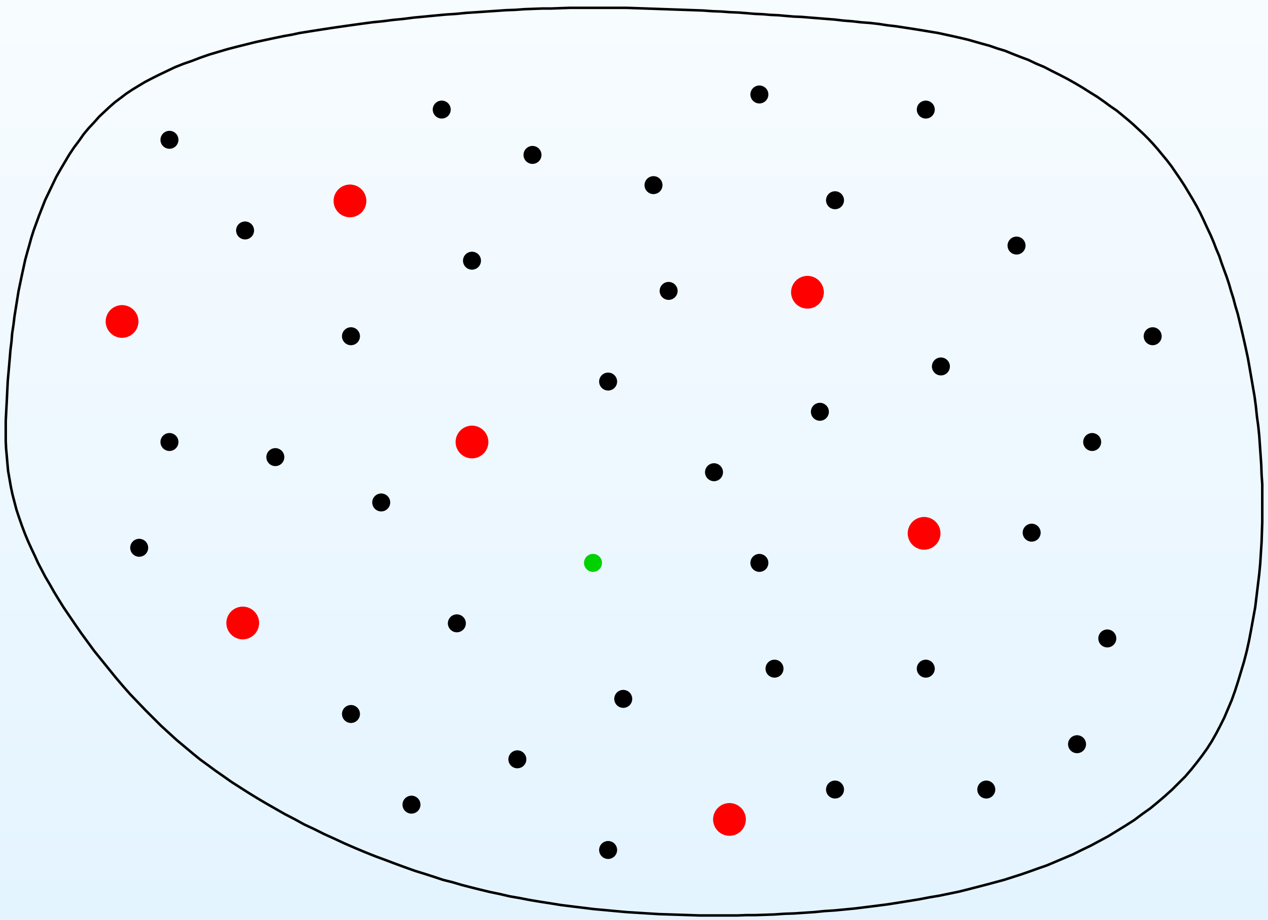

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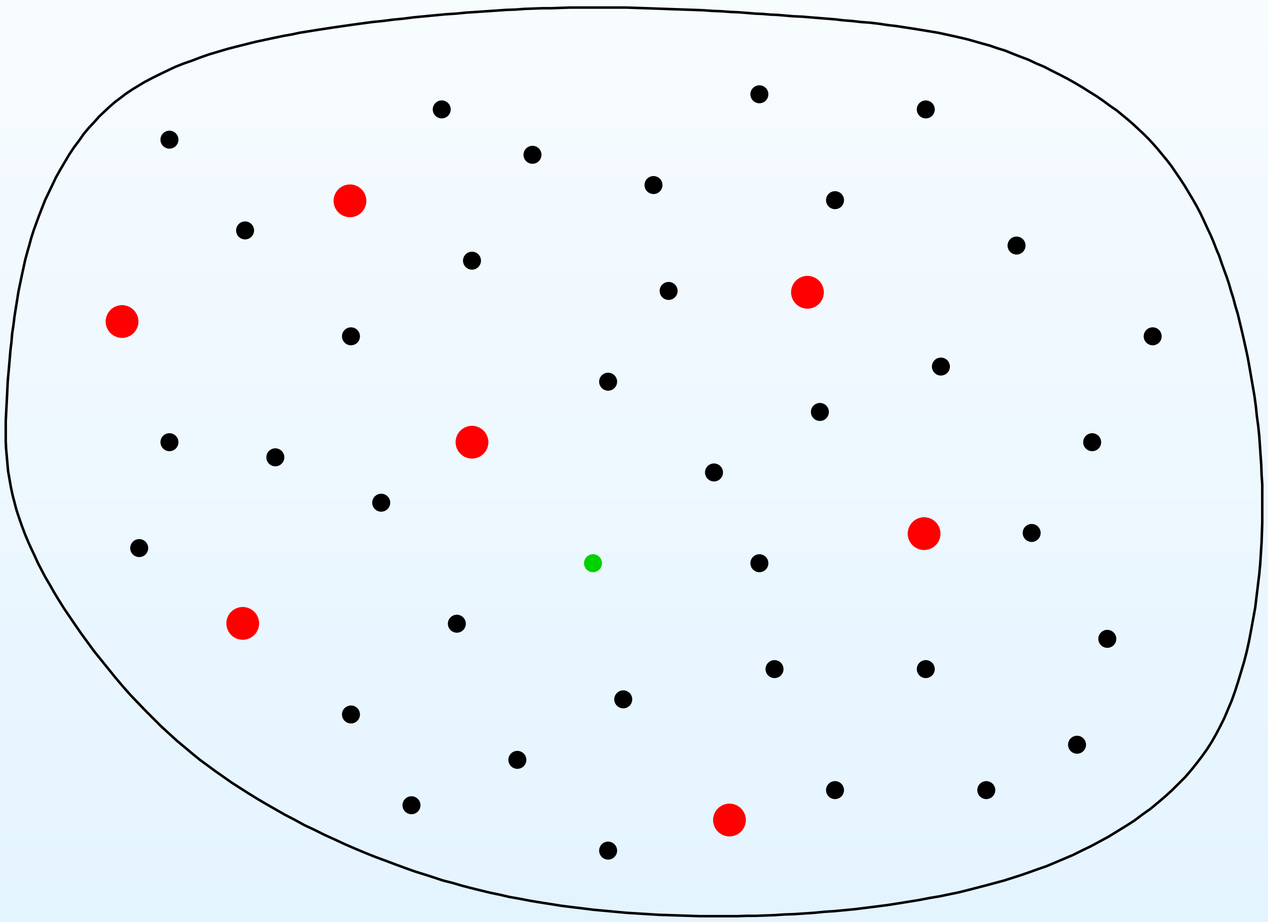

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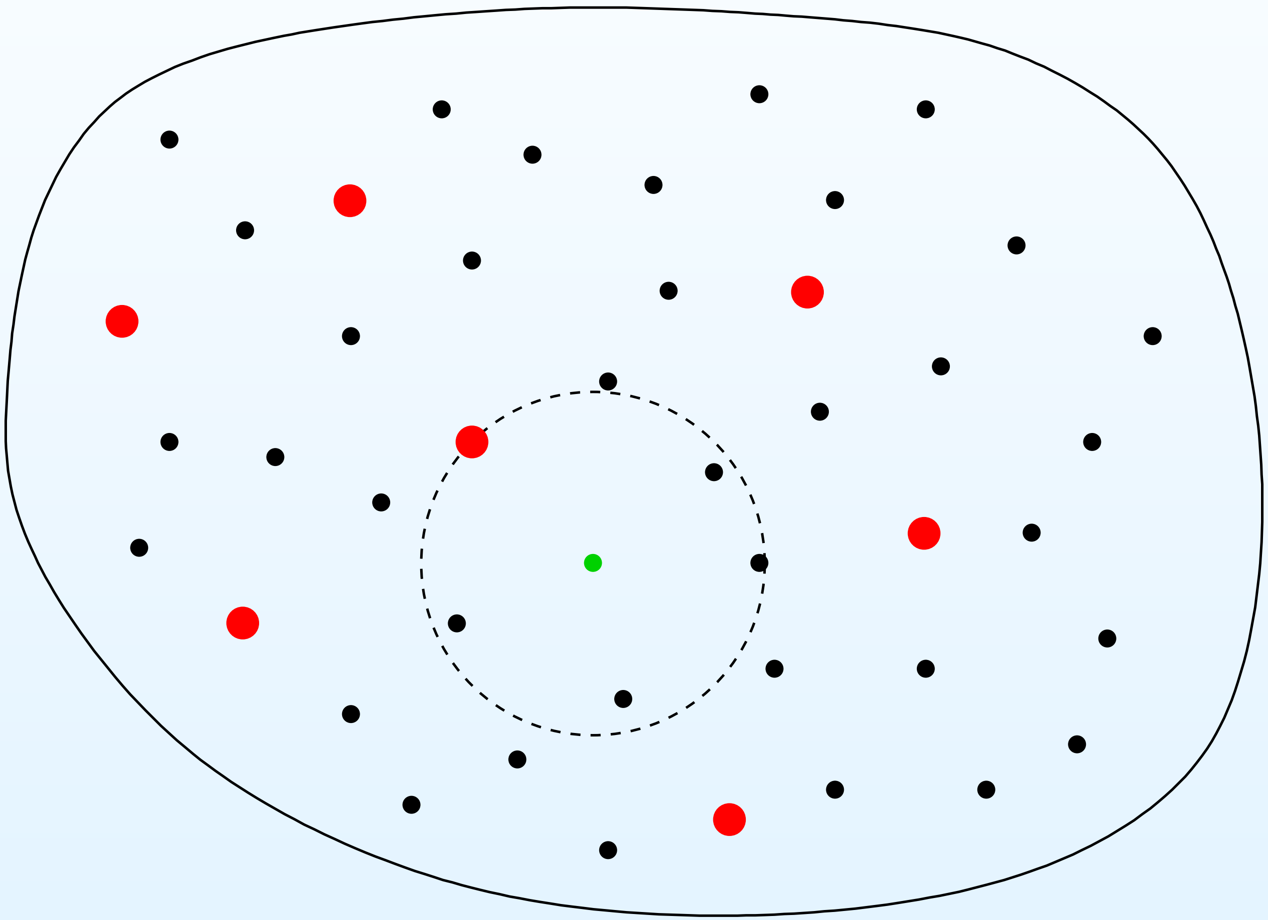

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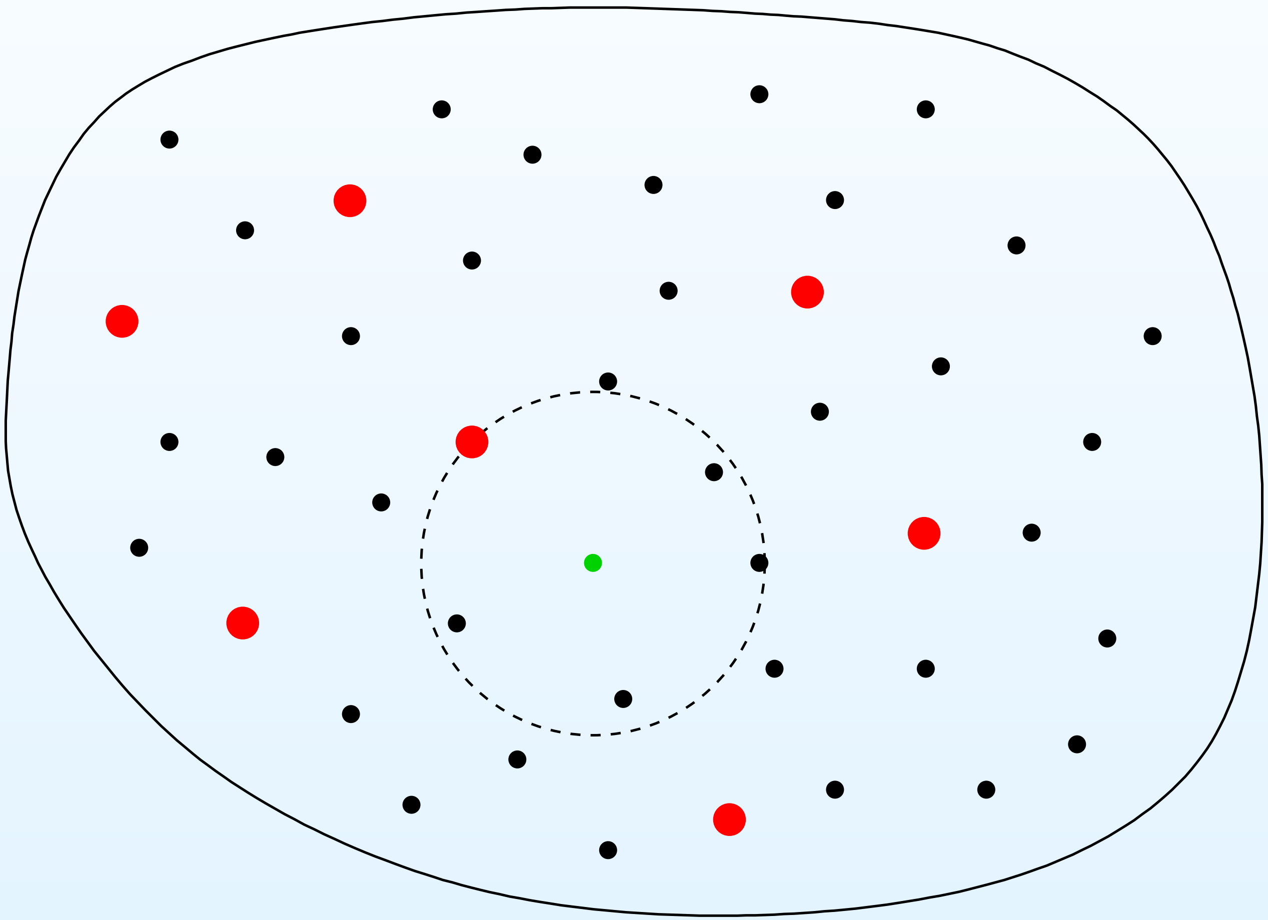

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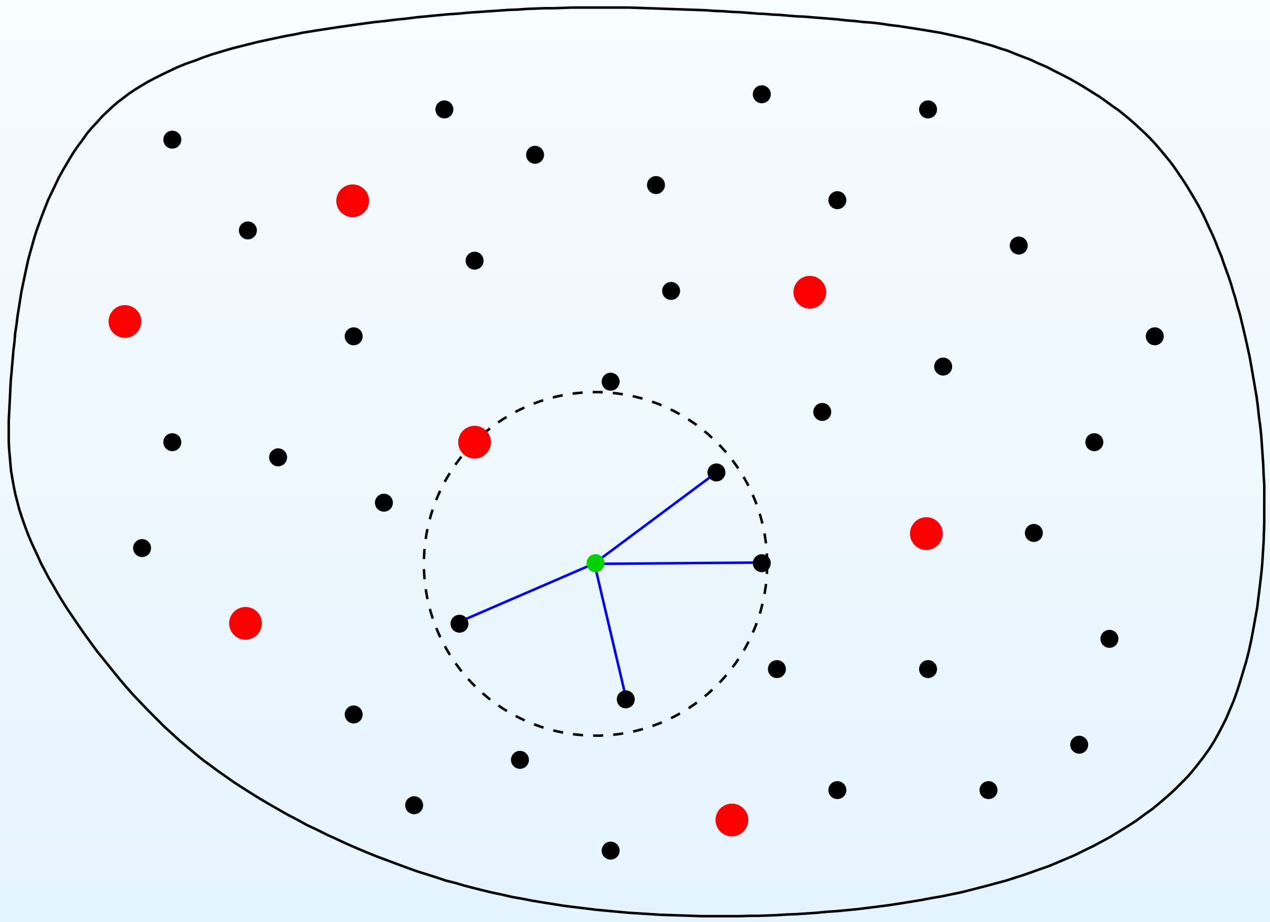

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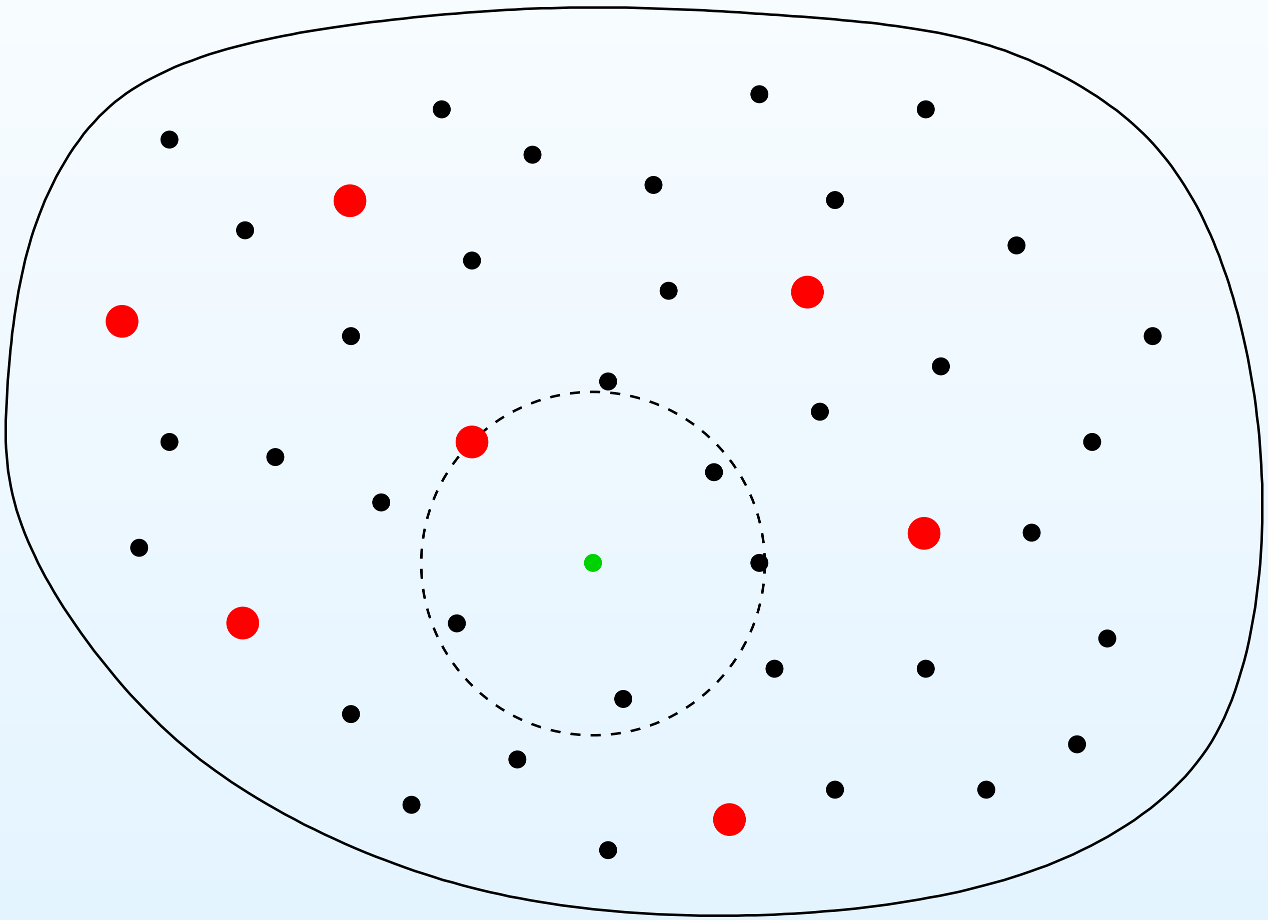

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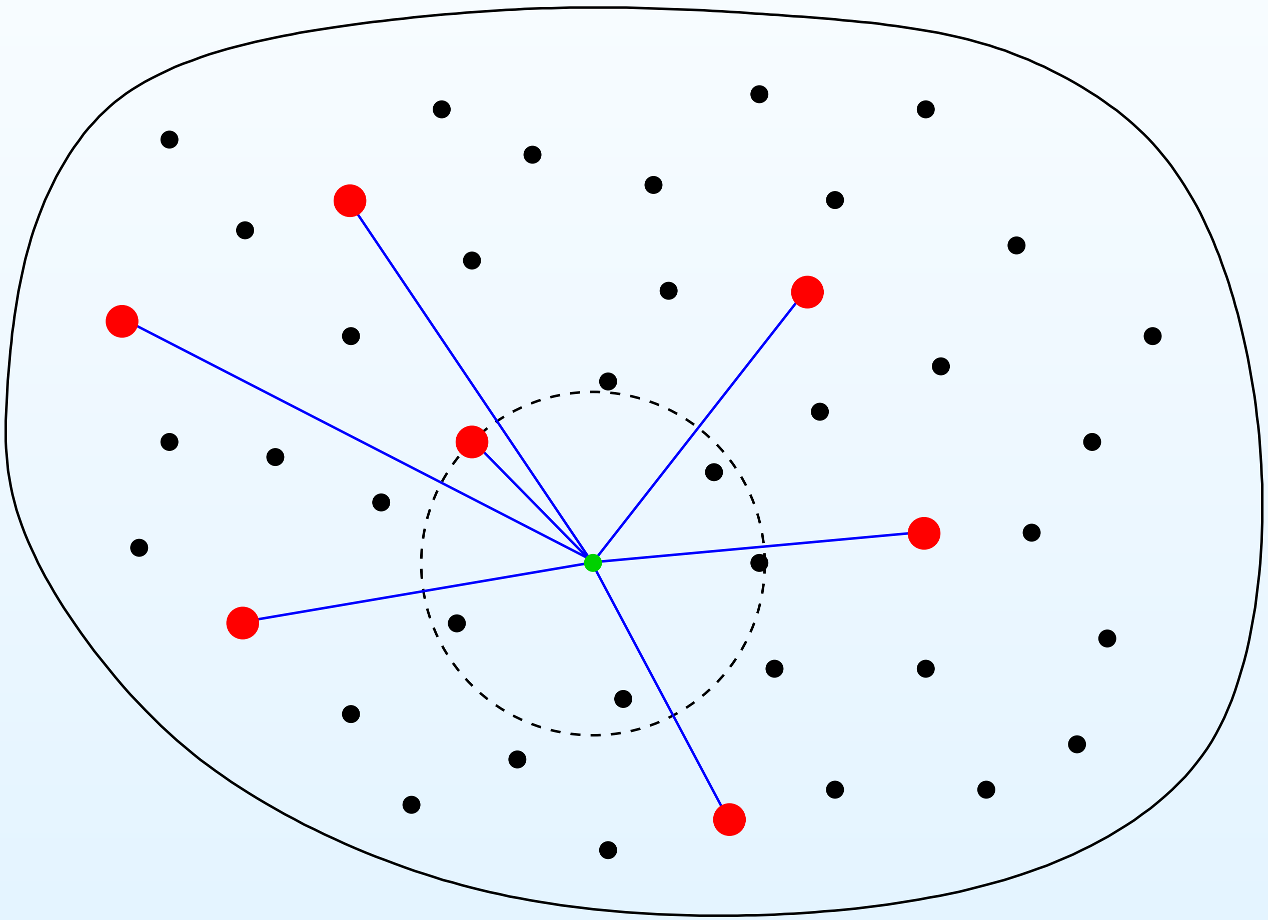

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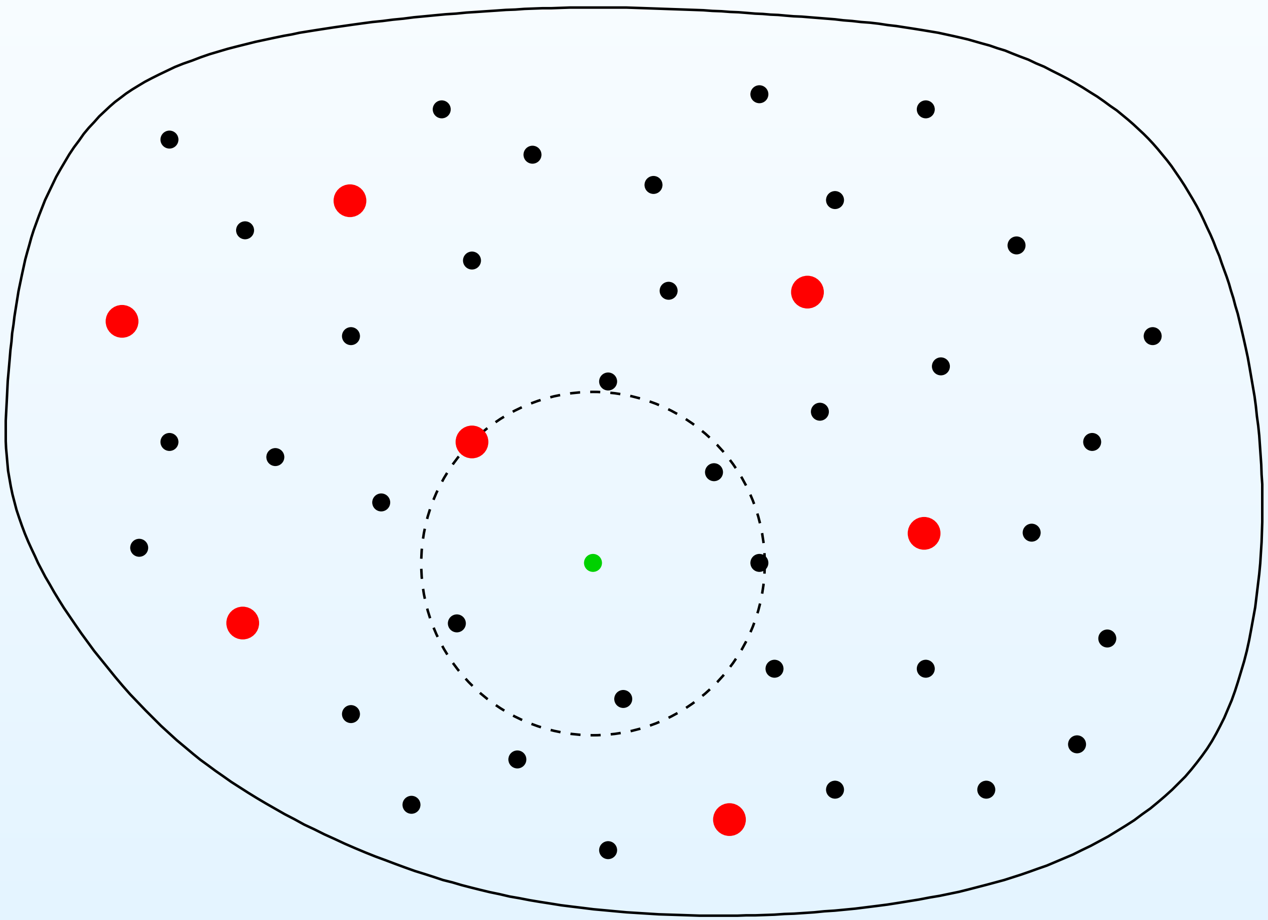

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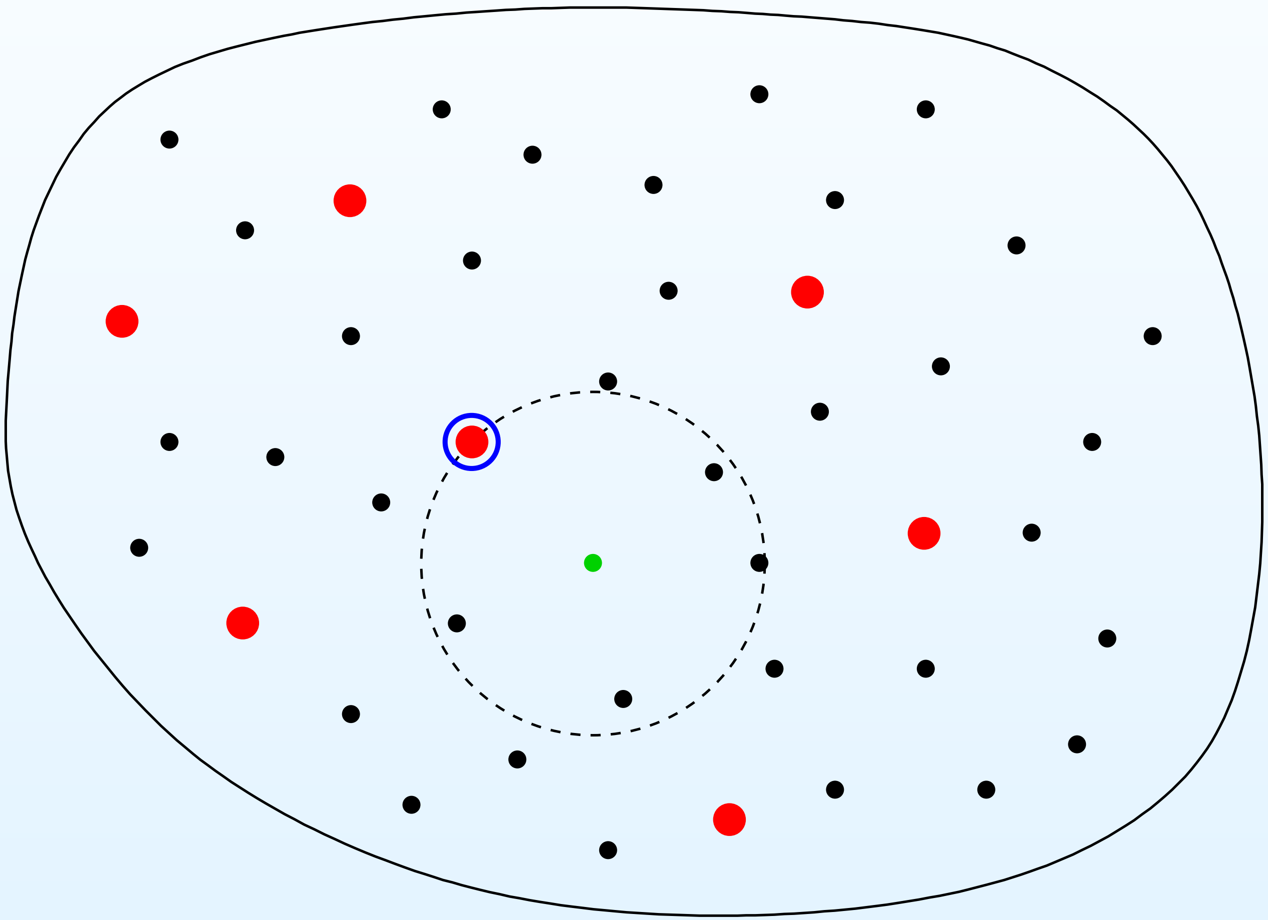

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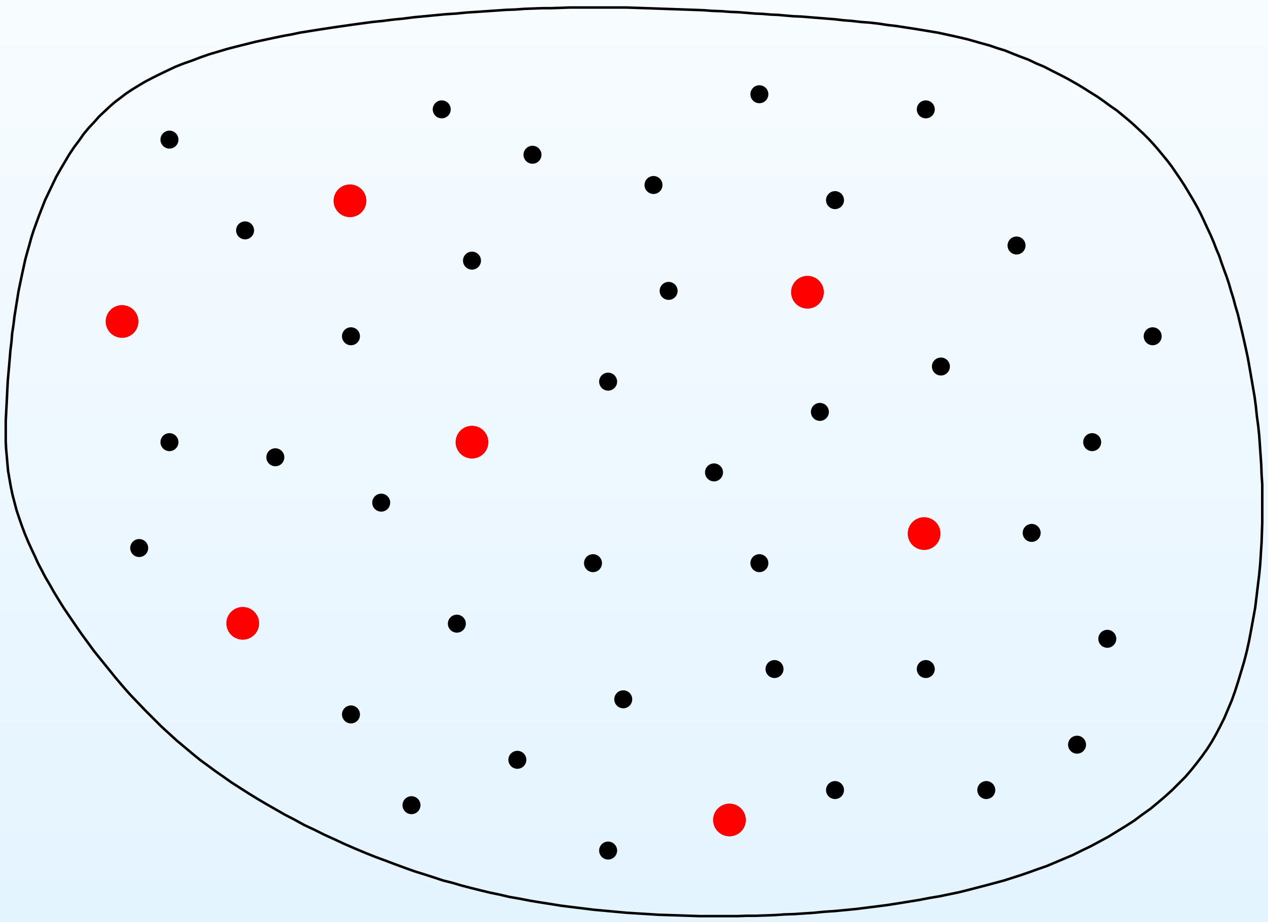

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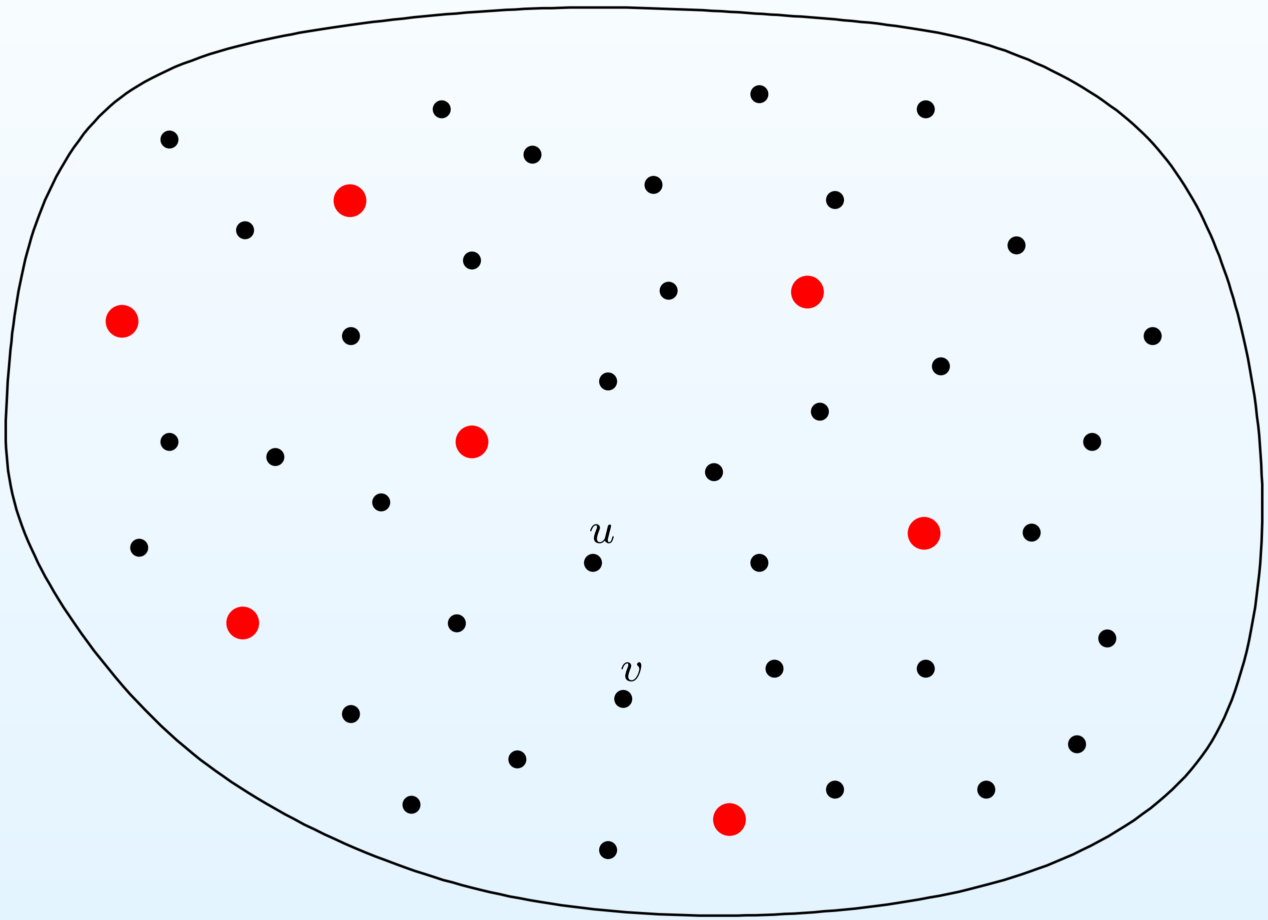

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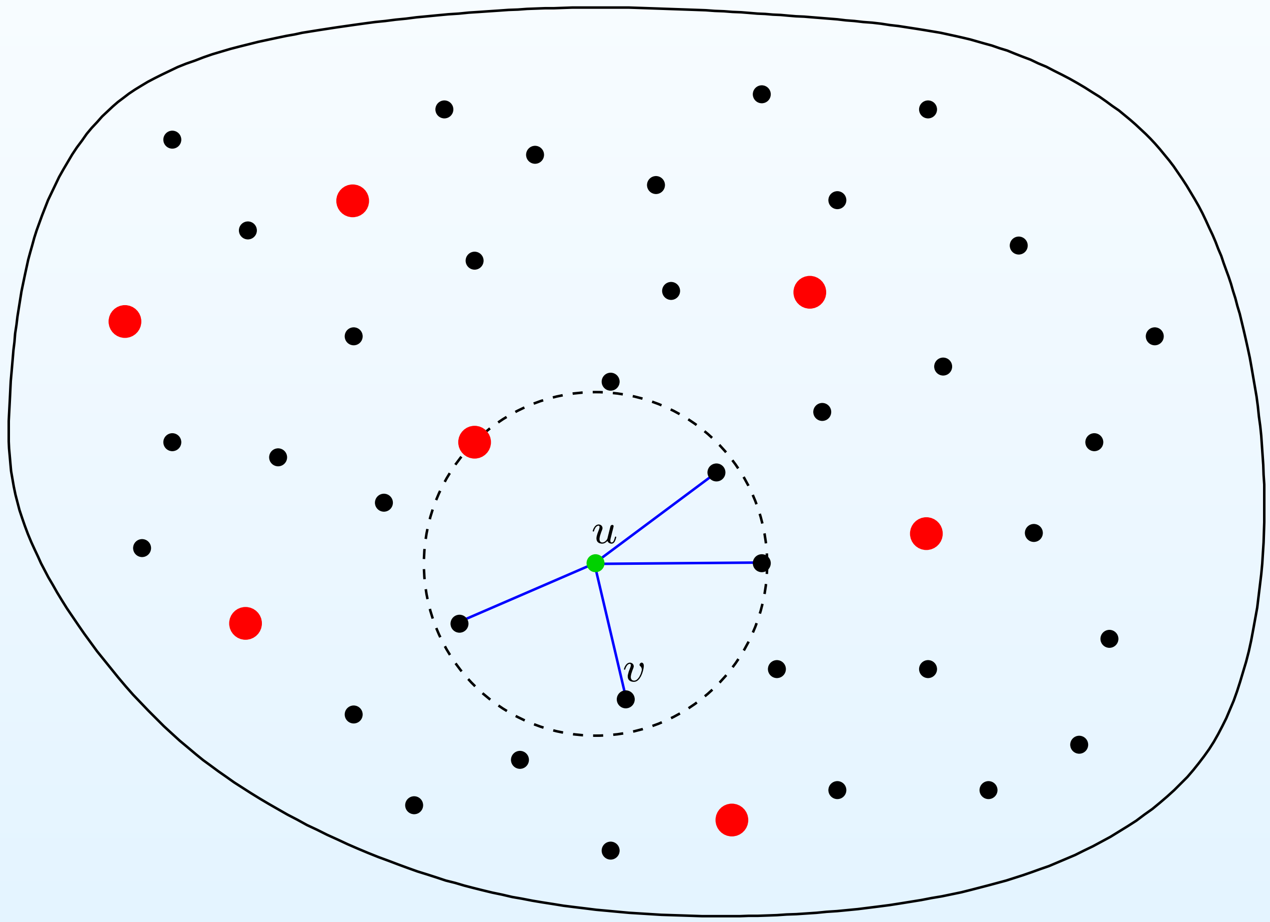

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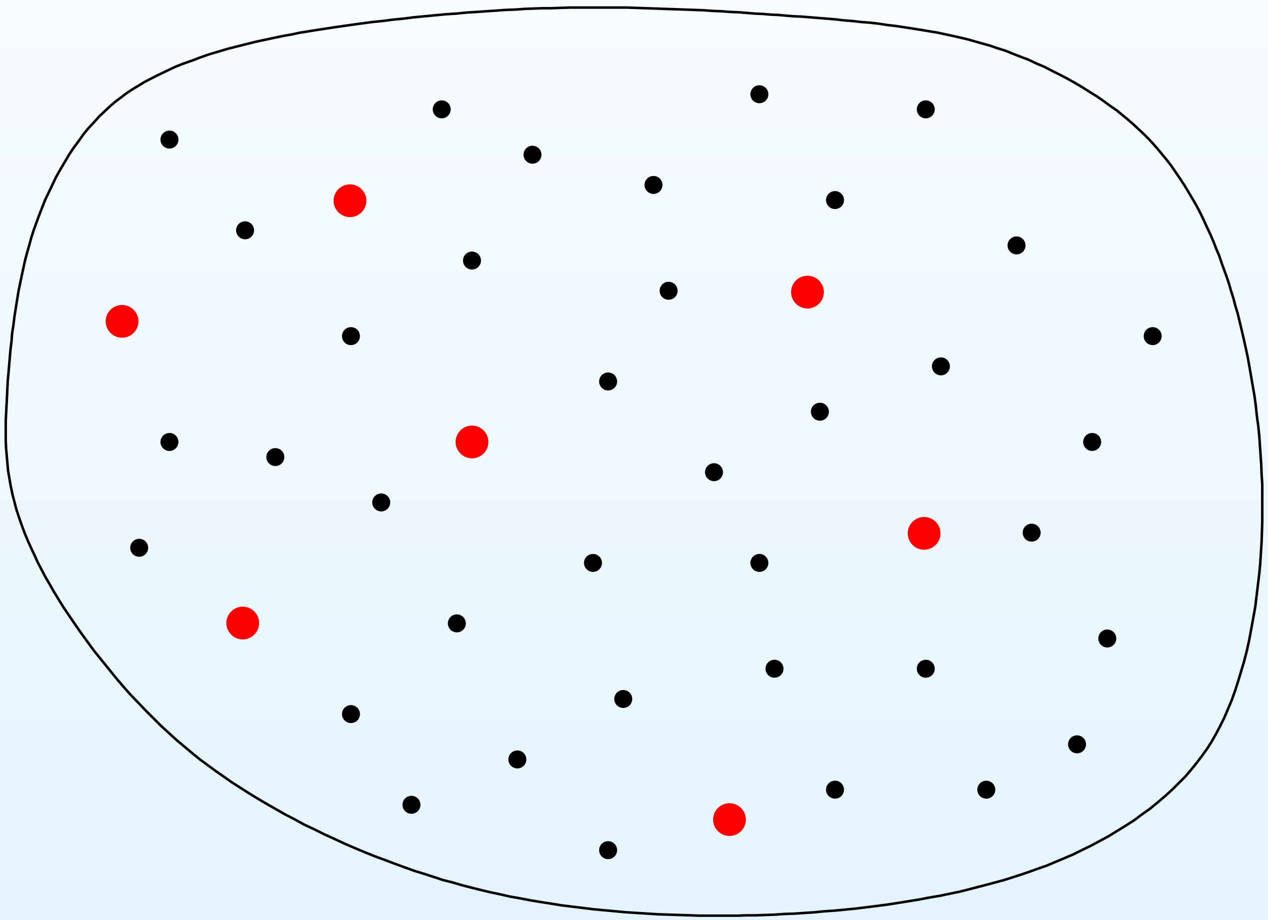

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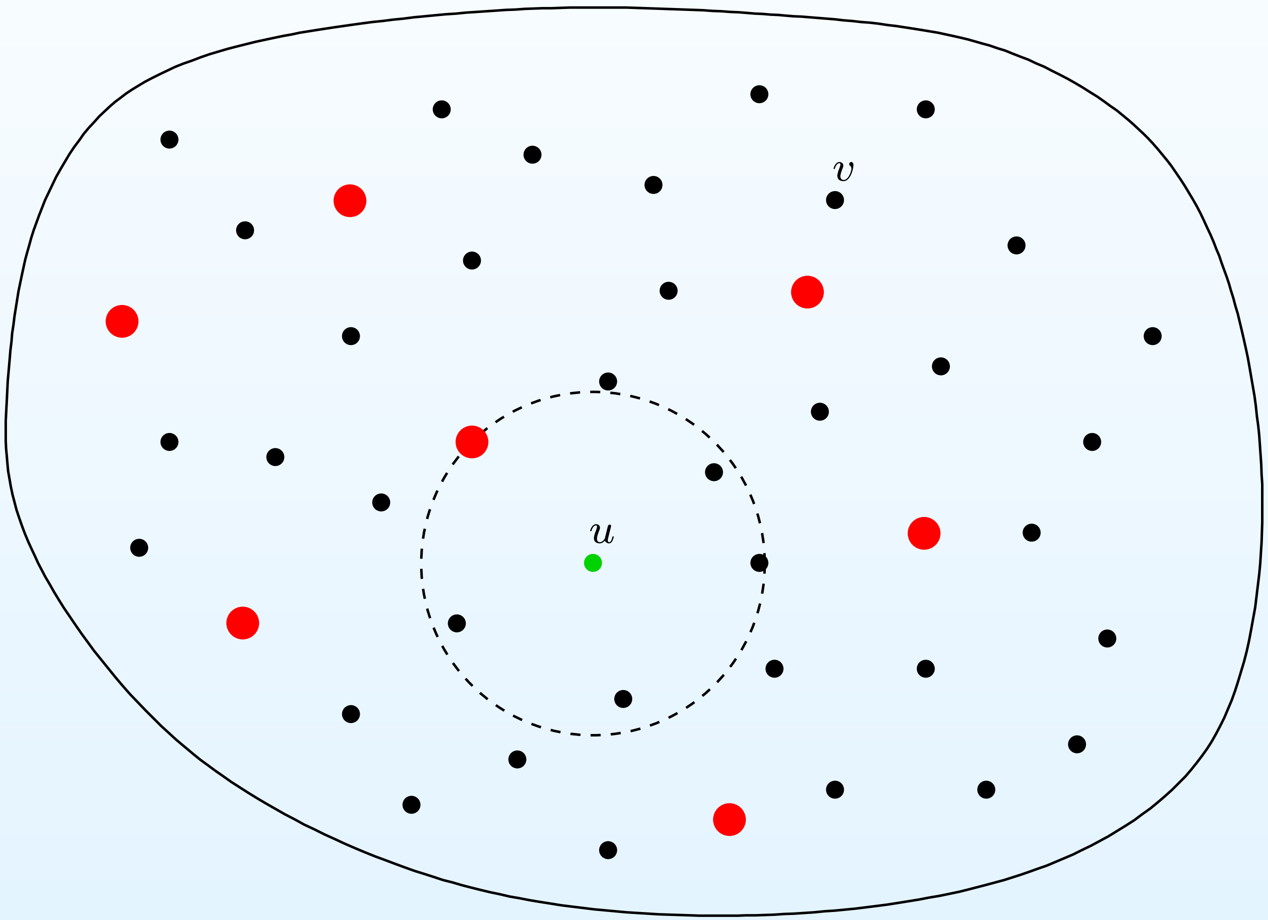
- Consider the opposite case:



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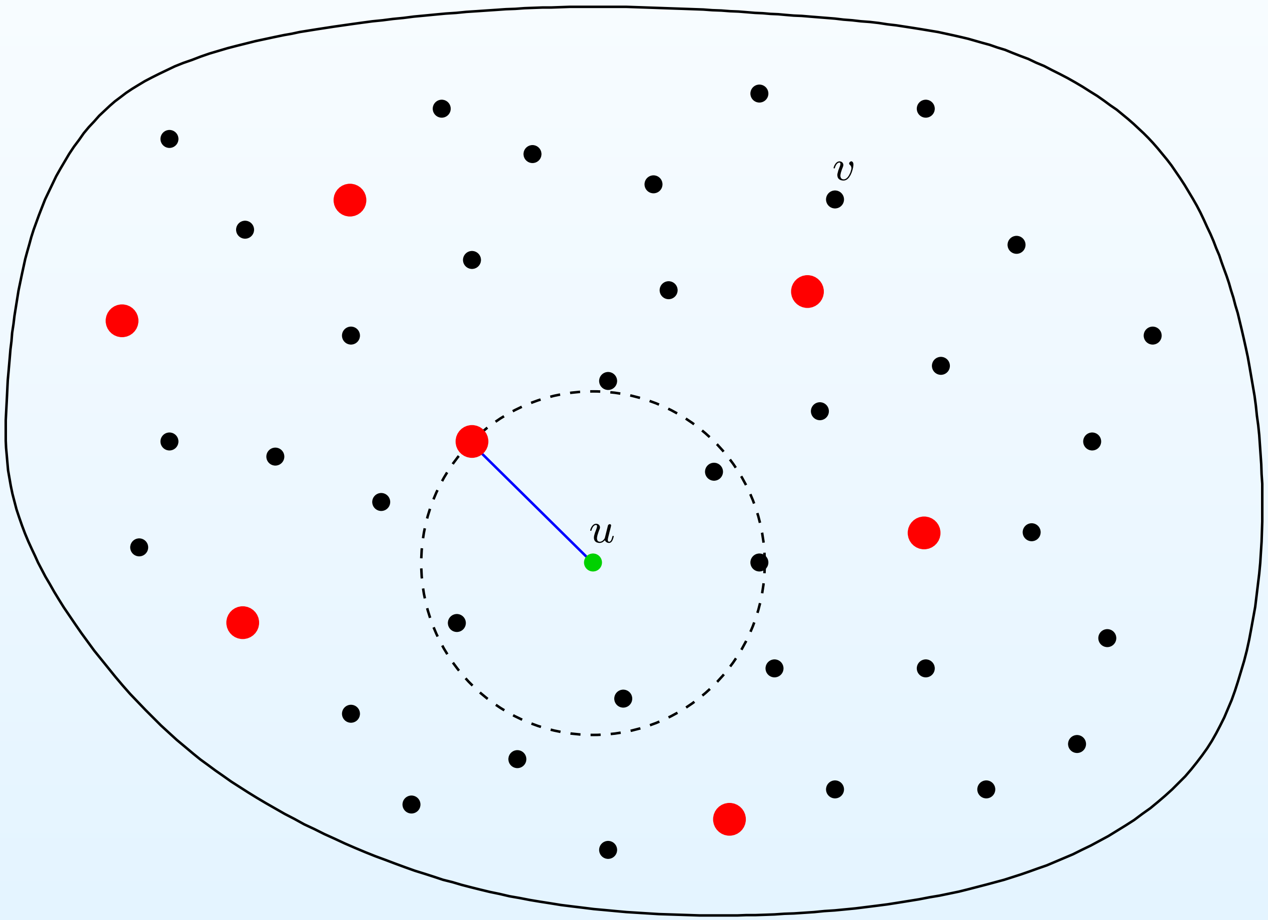
- We stored the nearest sample to u and the distance between them:



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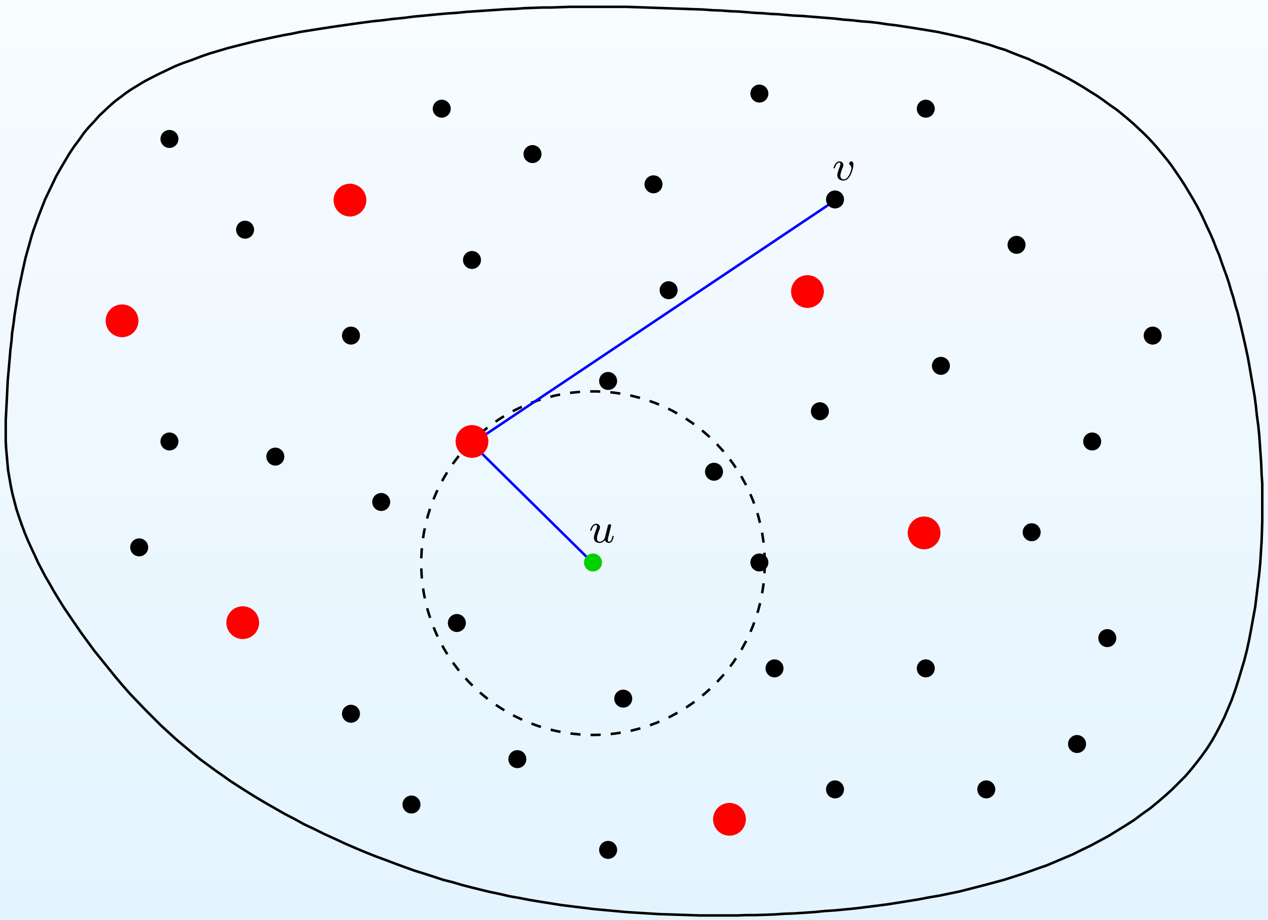
- And we stored the distance between v and this sampled vertex:



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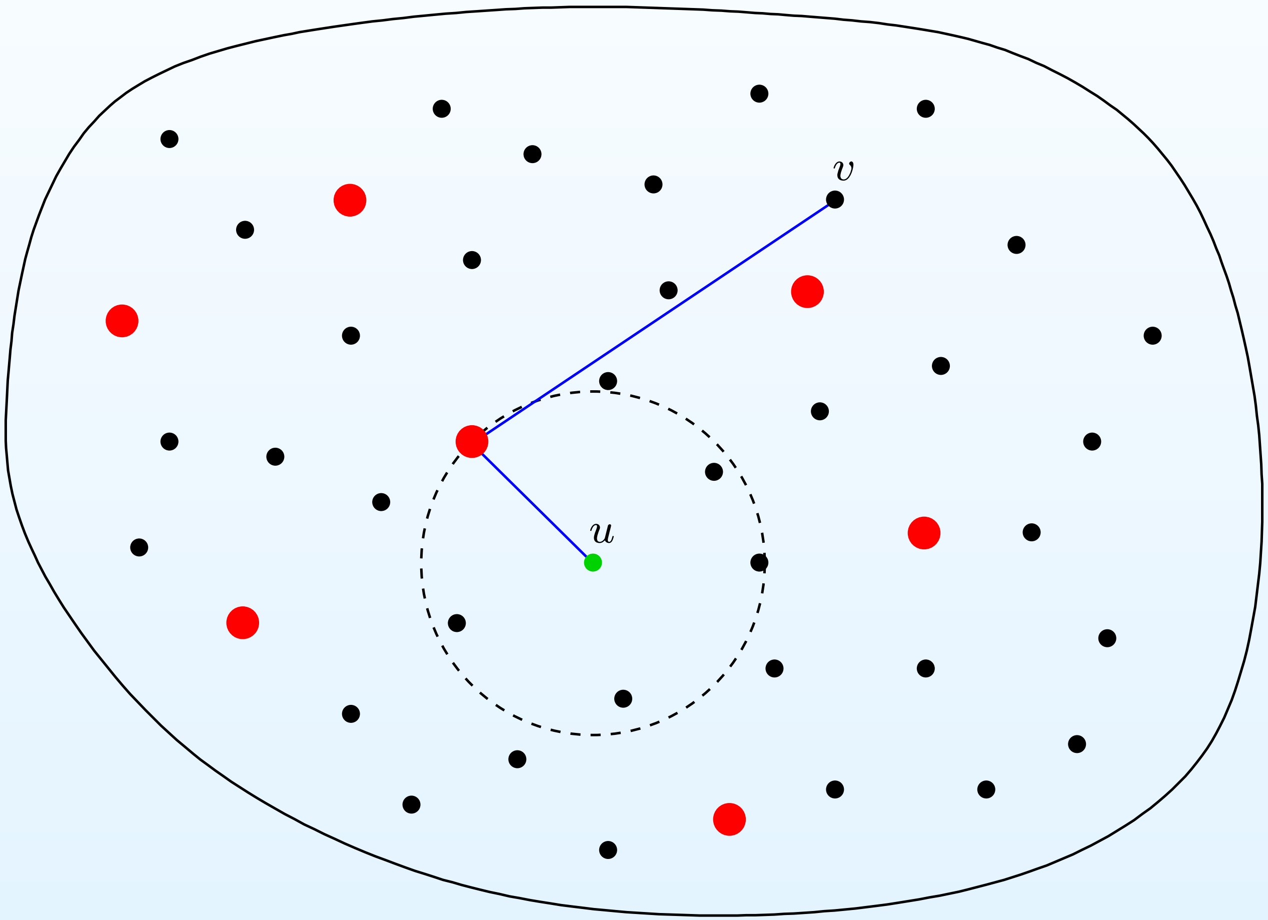
- The oracle returns the sum of these two distances:



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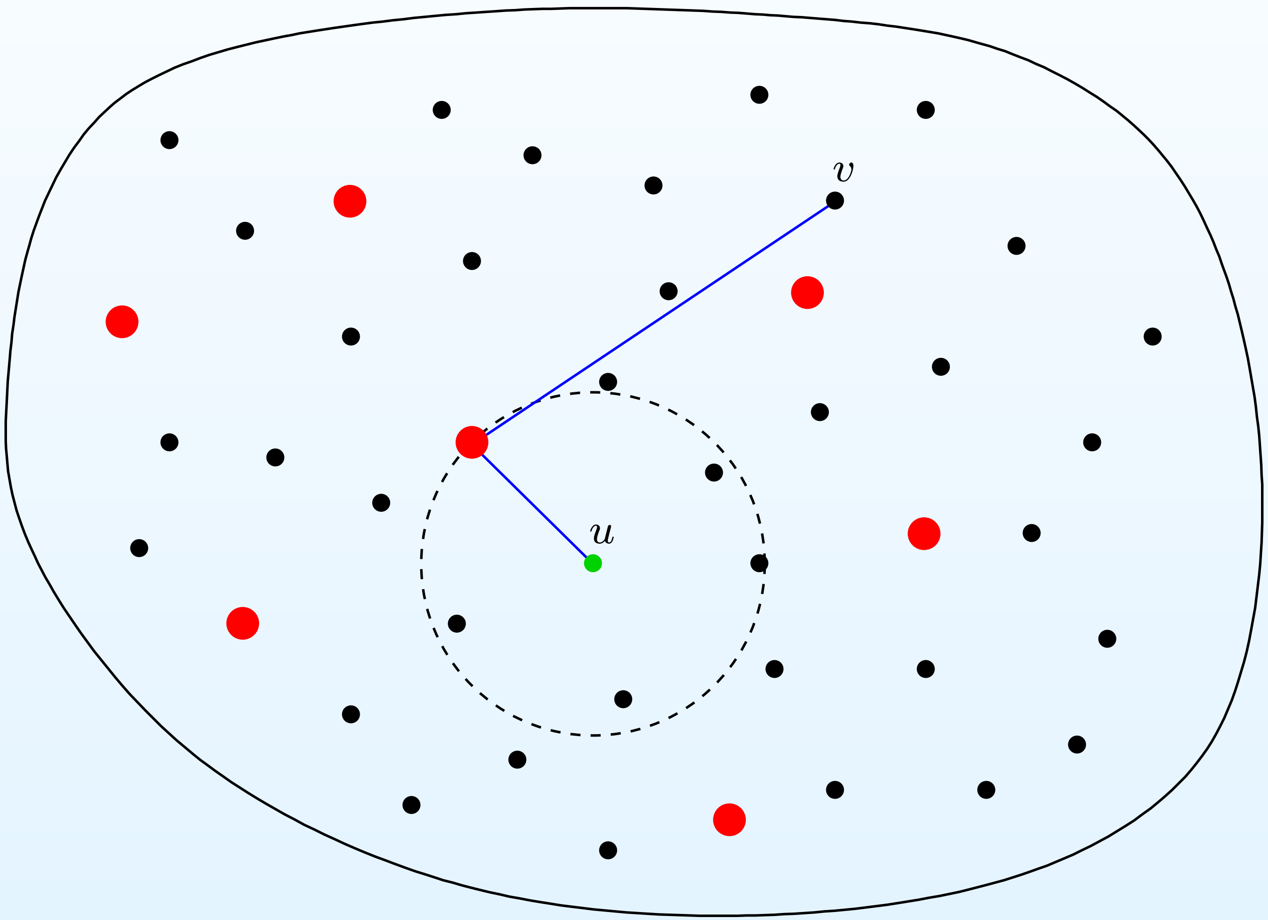
- This estimate has stretch 3 (exercise)



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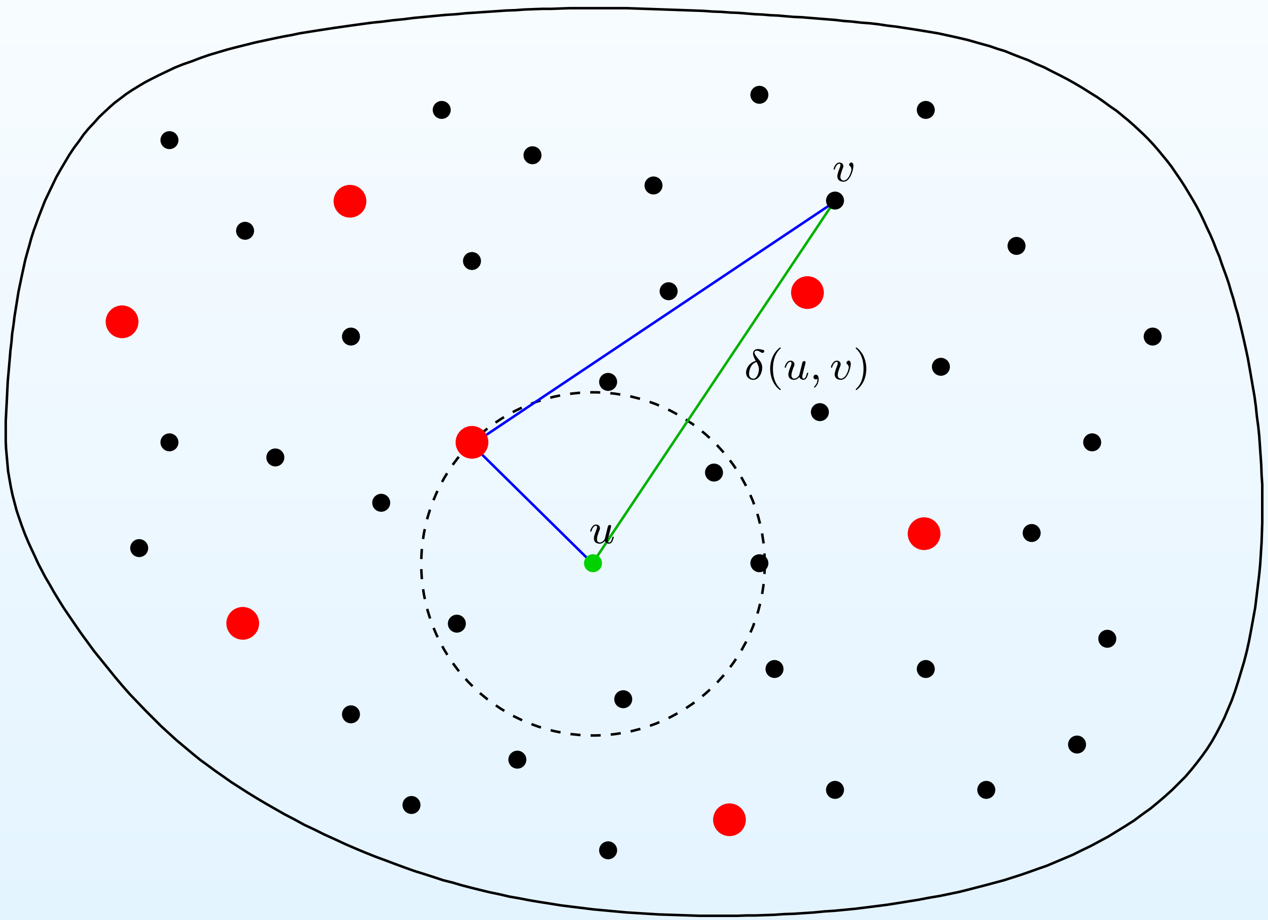

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  Part I: $\delta(s, v)$ for all $s \in S$ and all $v \in V$
  Part II: $\delta(u, v)$ for all $u \in V$ and all v closer to u than u's nearest vertex in S

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- Expected space for Part I is thus

$$E[|V||S|] = E[n|S|] = nE[|S|] = O(n^2 p)$$

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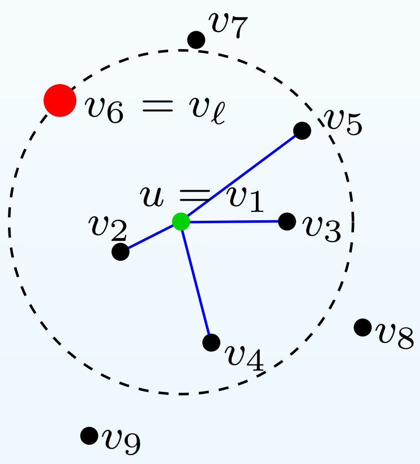

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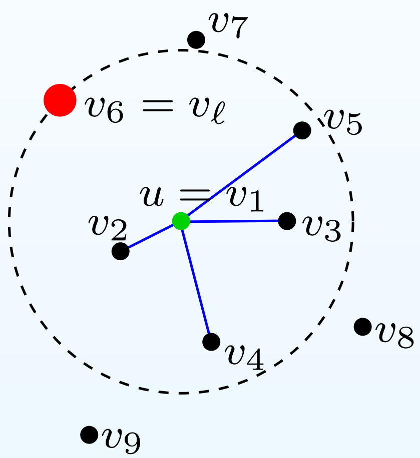

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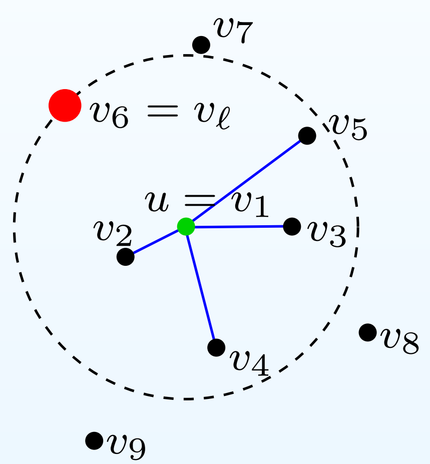

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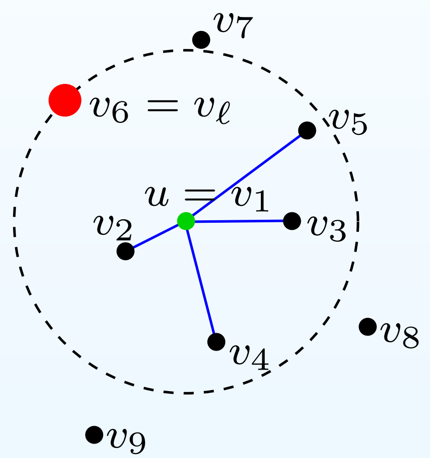

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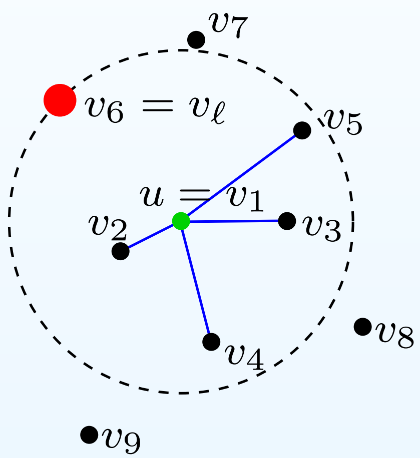

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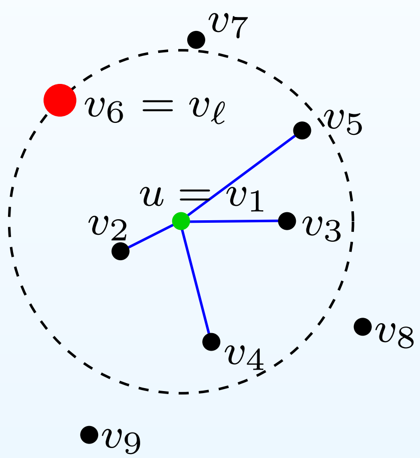

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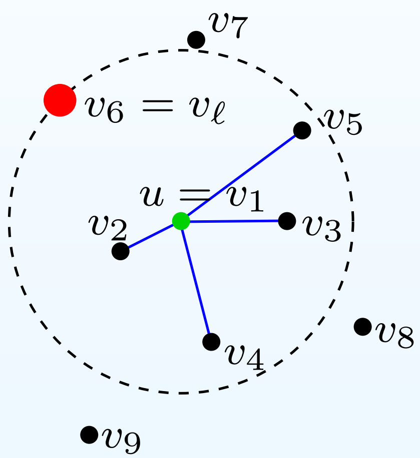

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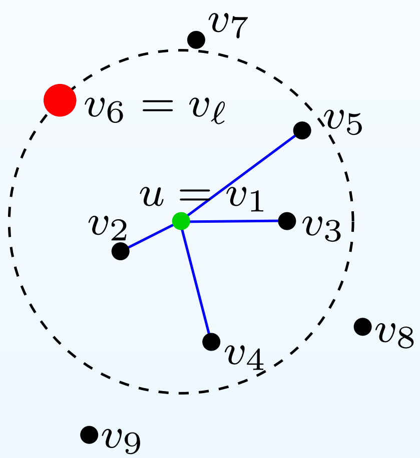

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- Expected space for Part II: $O(n \cdot E[\ell]) = O(n/p)$

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- Is this space bound measured in words or bits?

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- We now present and analyze this oracle

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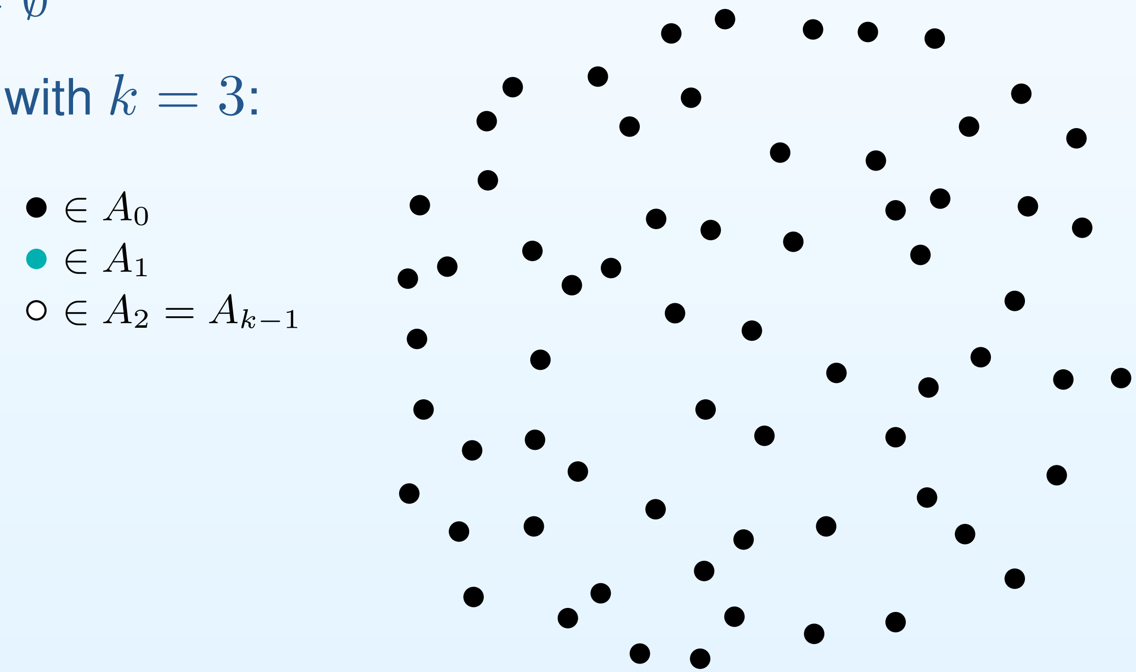

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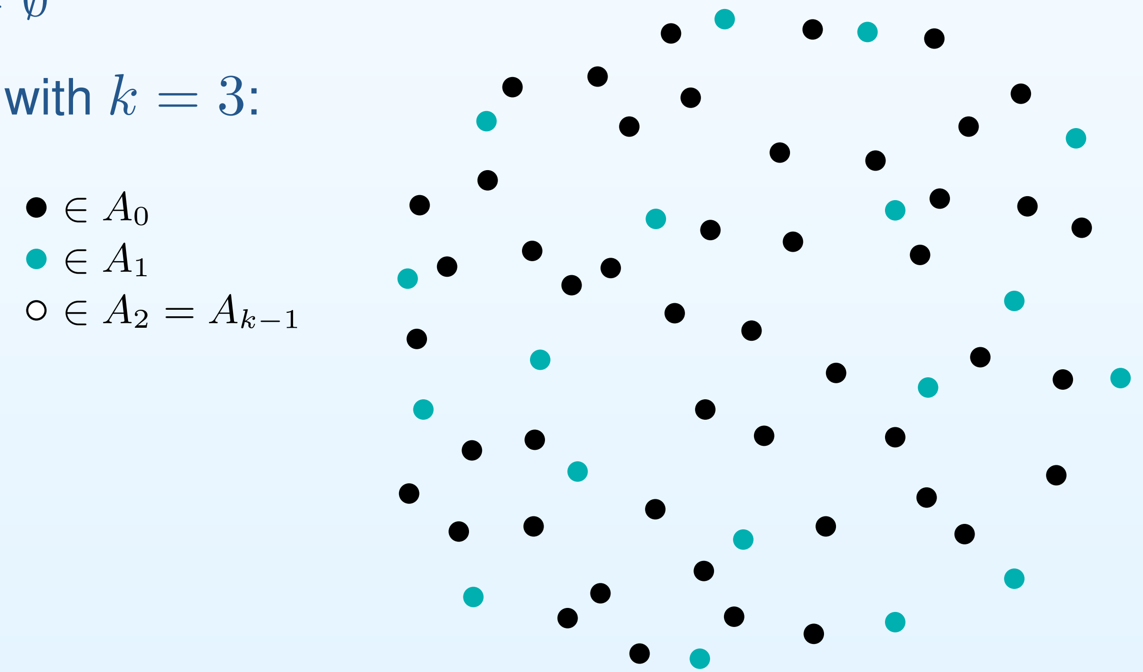

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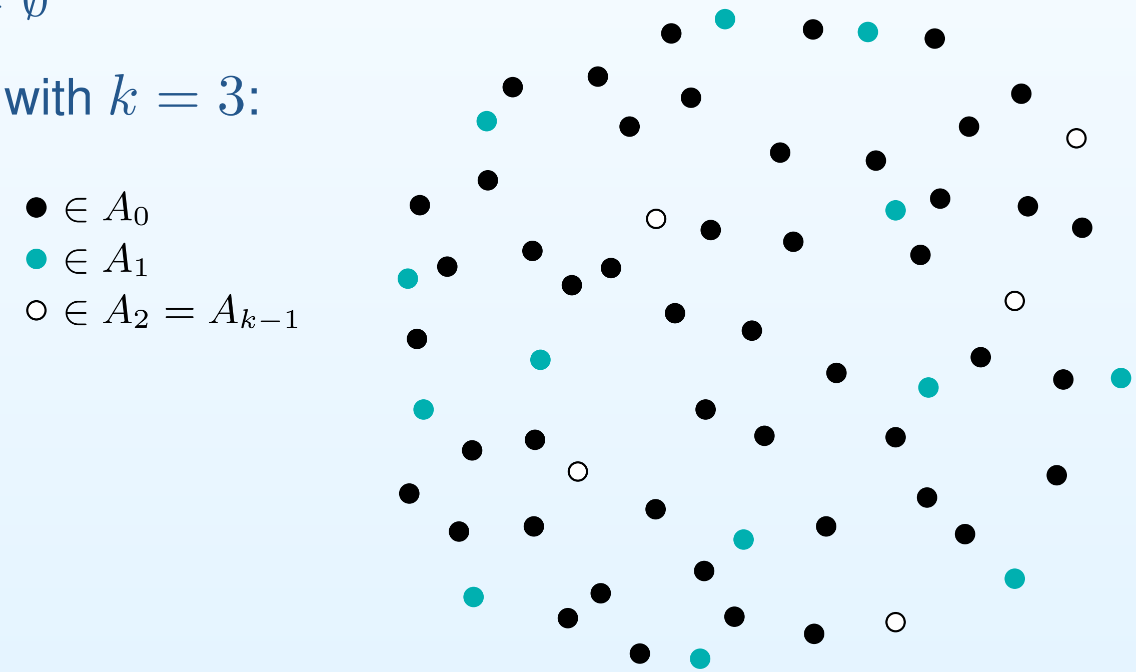

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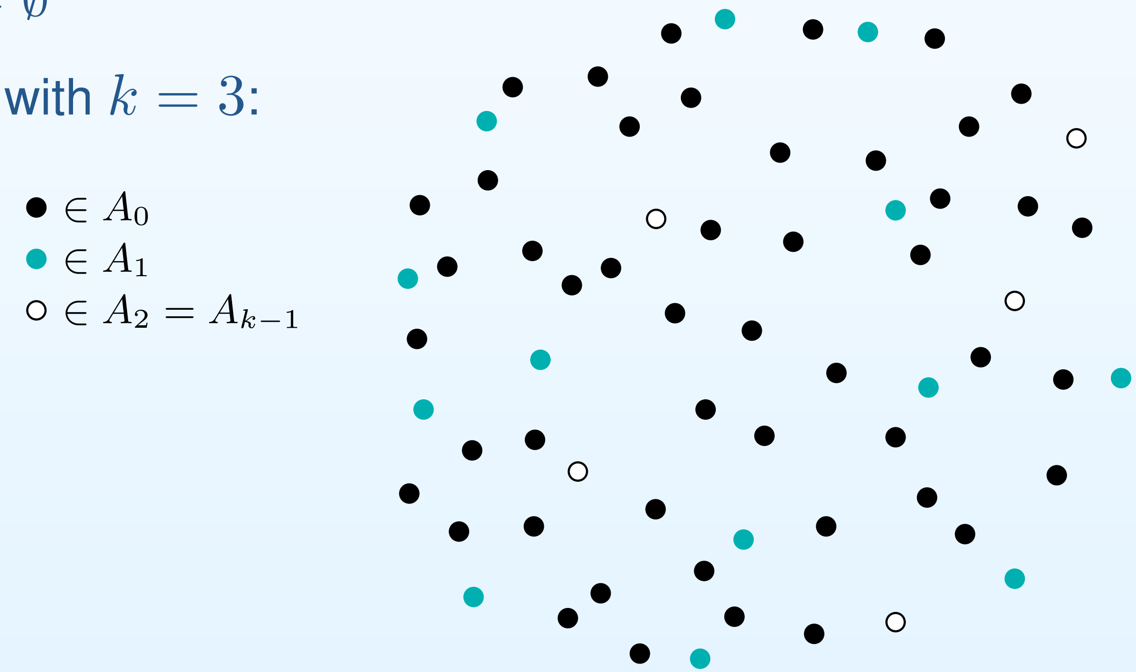

- Note that this matches what we did in the 3-approximate distance oracle where $k = 2$

Distance to a Set

- For $u \in V$ and $i = 0, \ldots, k$, define

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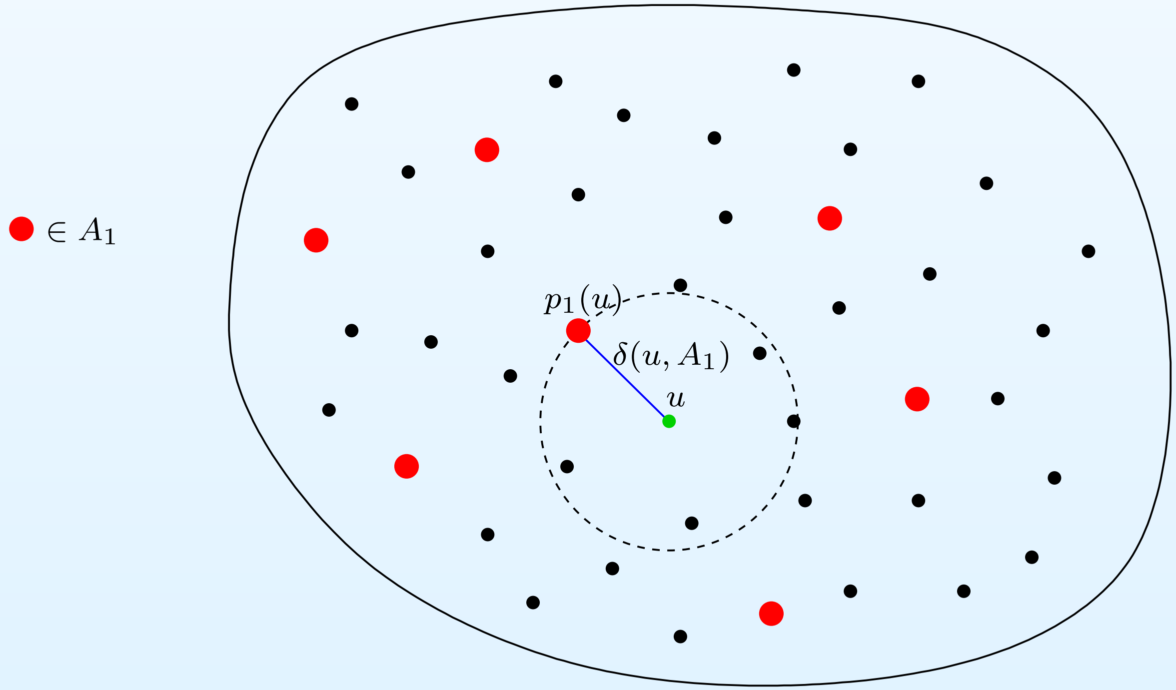

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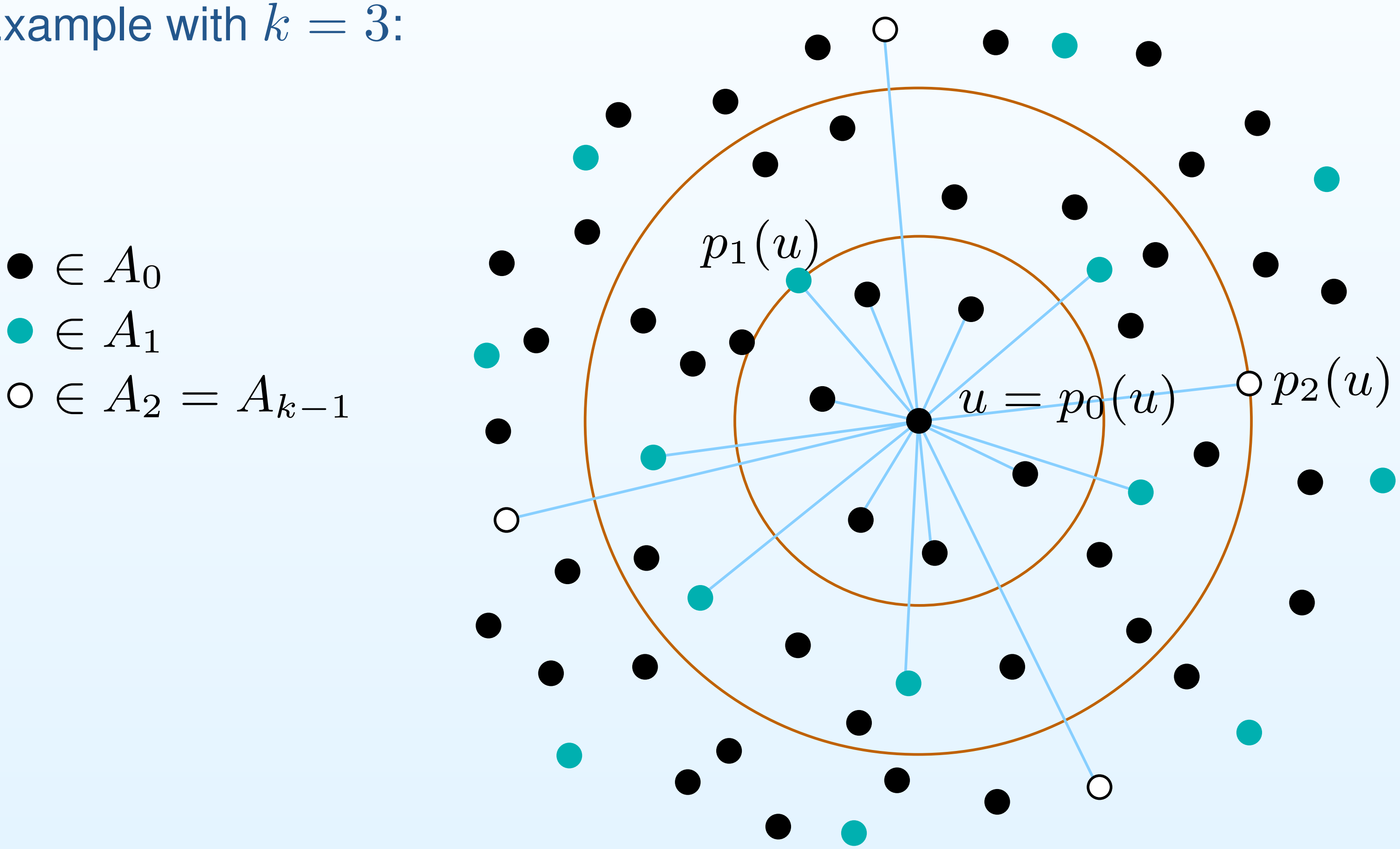

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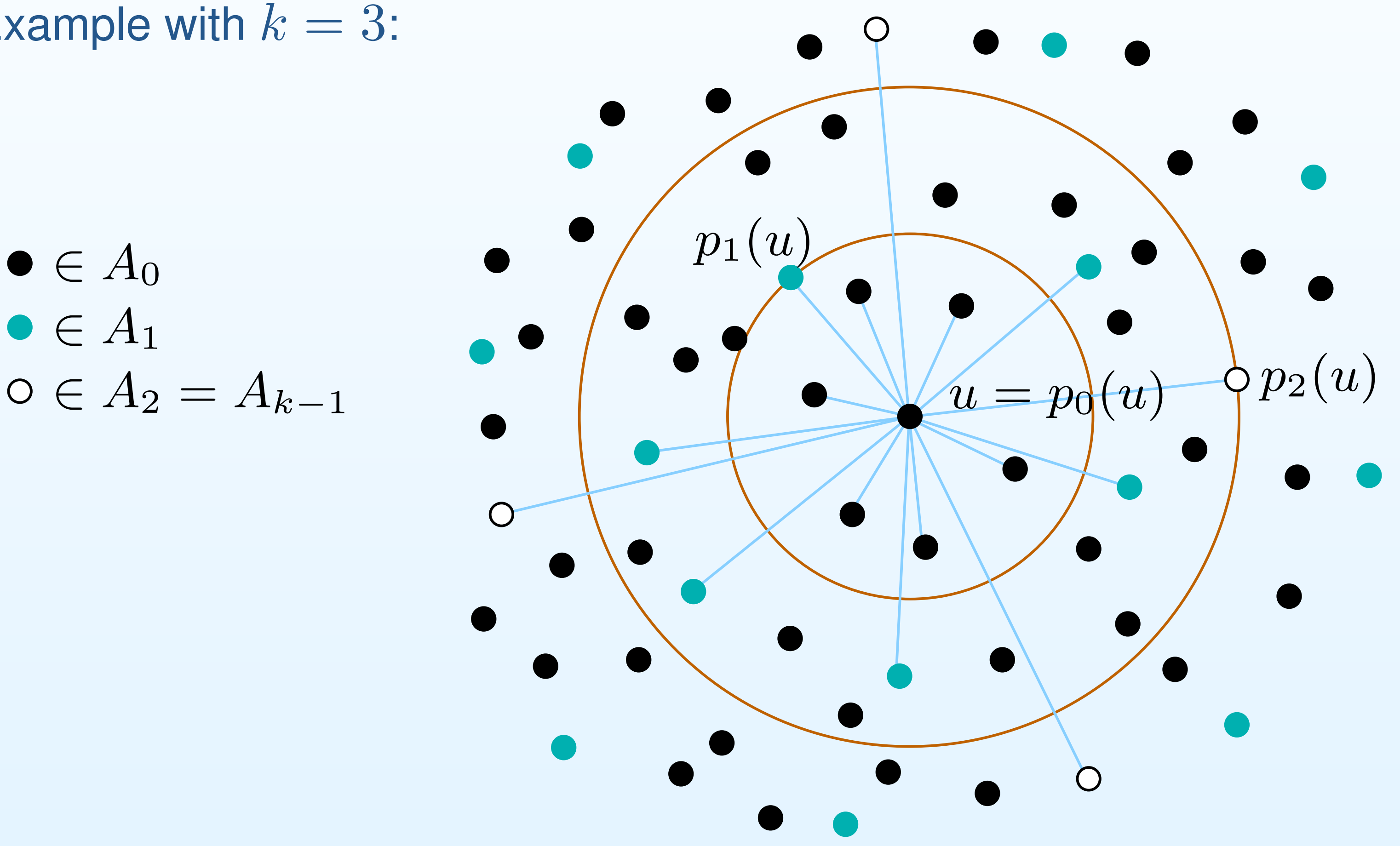

- Observe $A_k = \emptyset \Rightarrow \delta(u, A_k) = \infty \Rightarrow A_{k-1} \subseteq B(u)$

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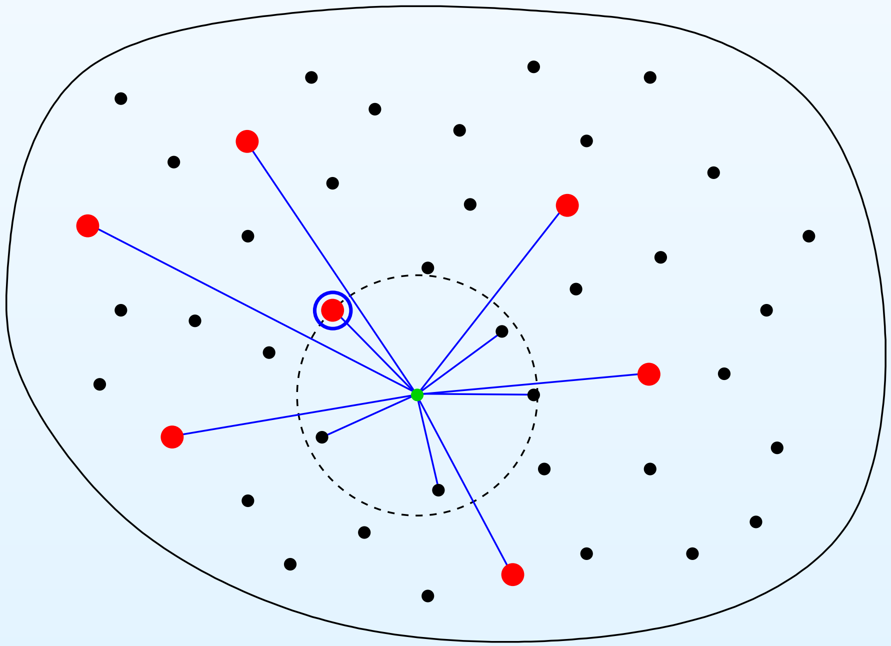

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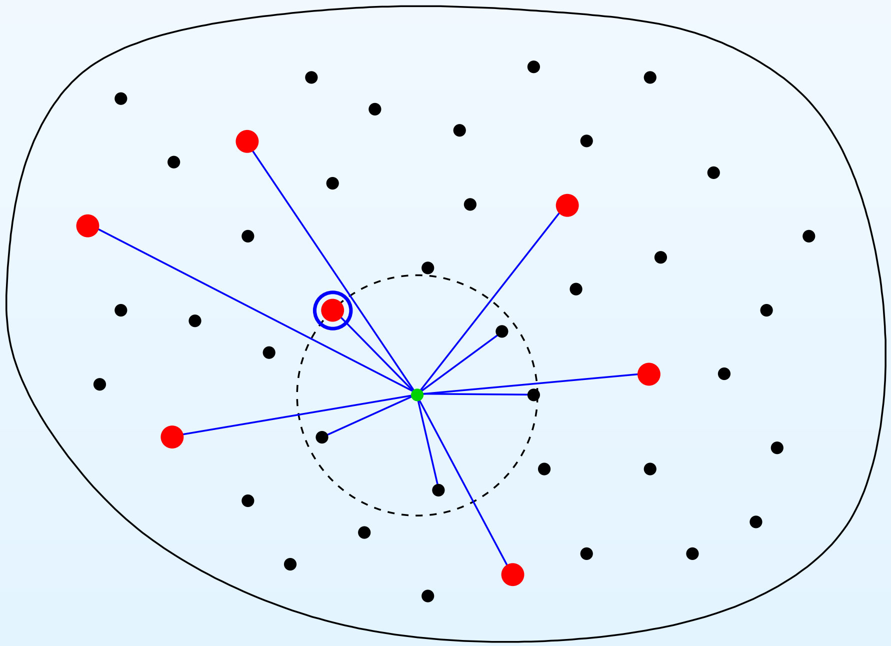

- Space for oracle is bounded by total space for all bunches

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- By linearity of expectation, the total expected space is

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- Recall: p is the probability that a vertex in A_i is sampled and included in A_{i+1}, $i = 0, \ldots, k-2$

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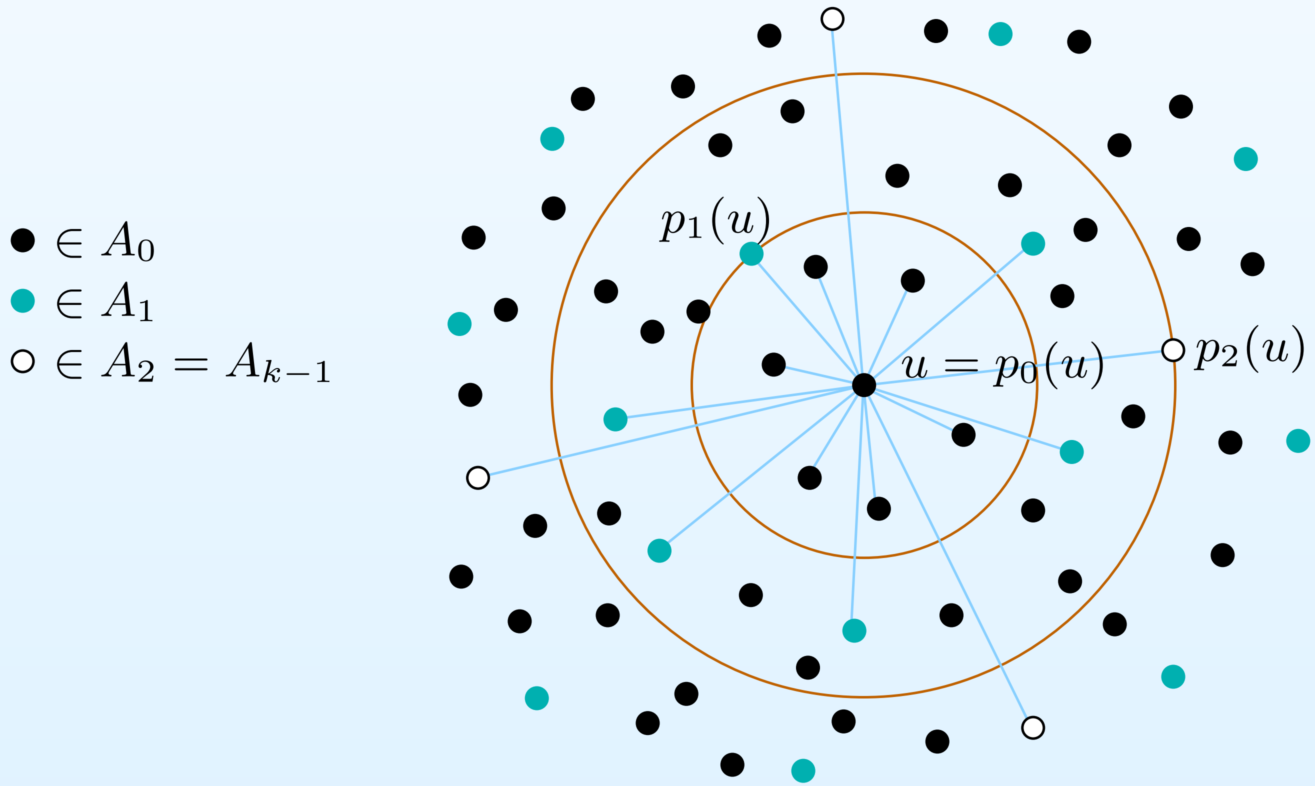

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- Thus, $E[|B(u)|] = O(k/p + np^{k-1})$ as desired

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w \leftarrow u,\ i \leftarrow 0 \\
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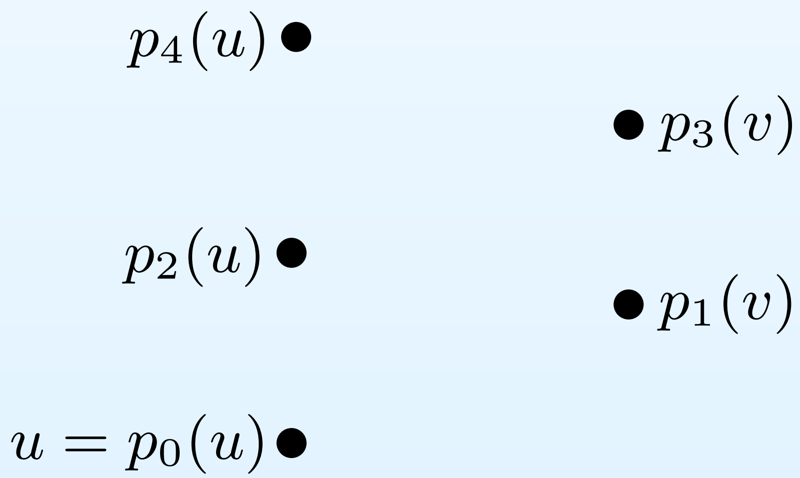

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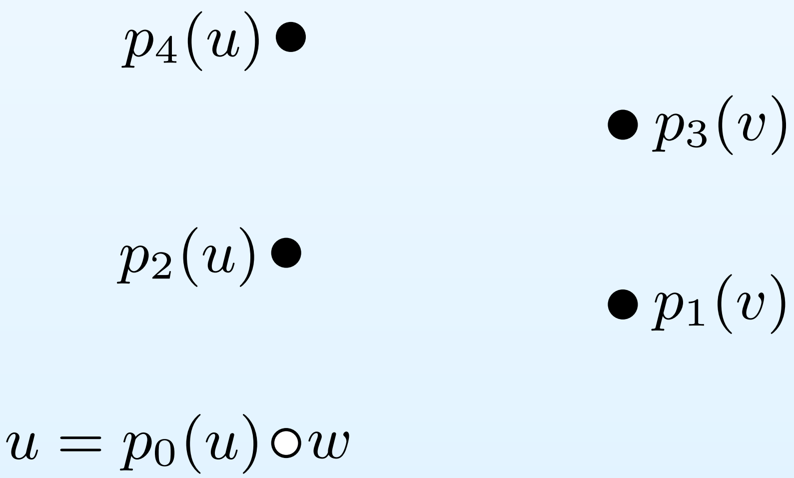

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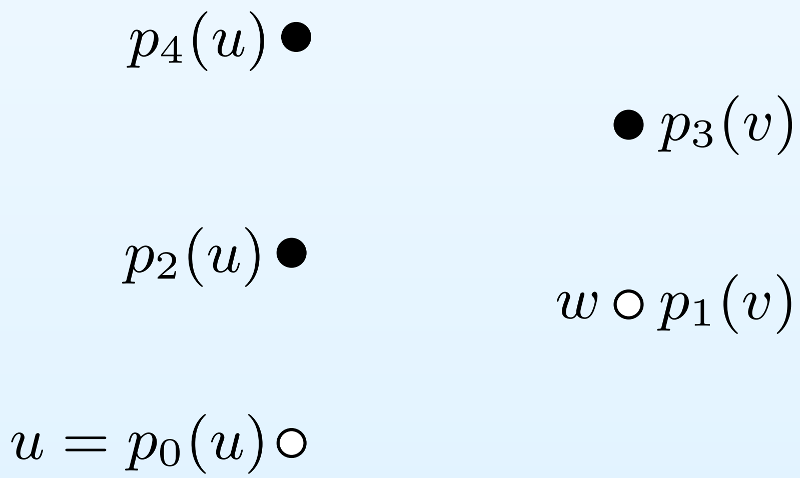

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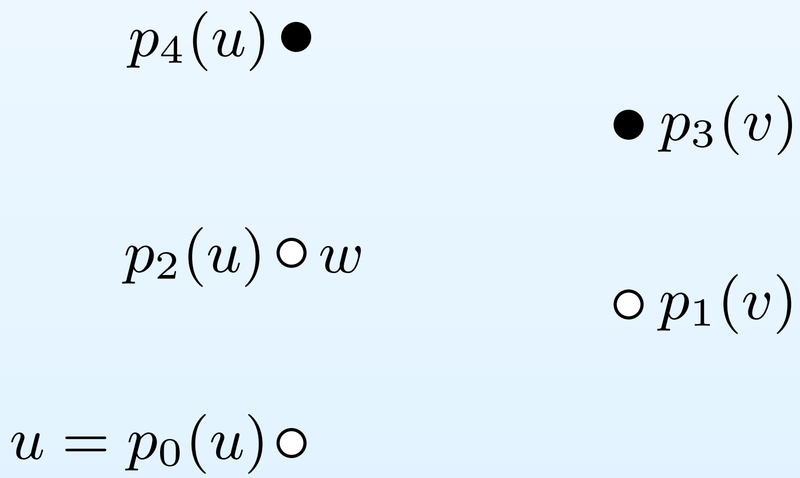

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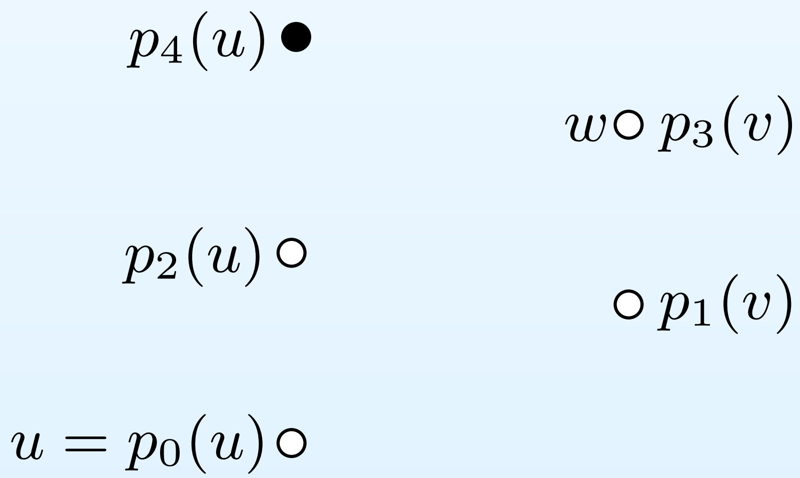

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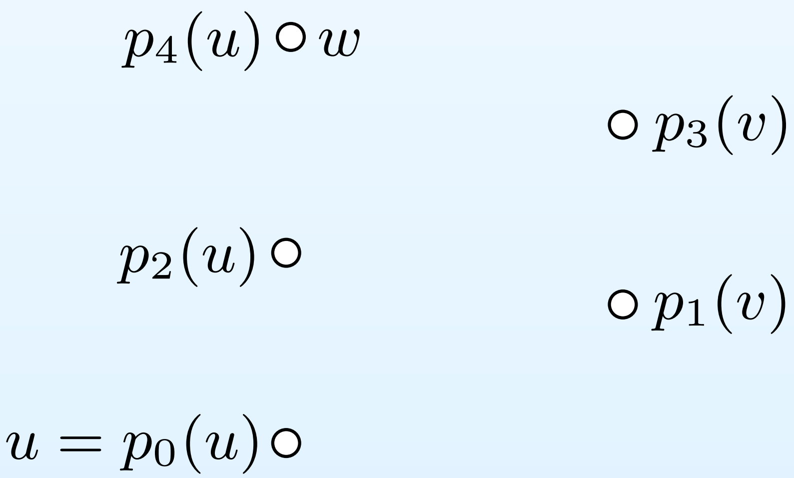

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  - Show that in Line 6, both distances $\delta(w, u)$ and $\delta(w, v)$ are stored by the data structure

Bounding the increase in stretch per iteration

- We will show $\delta(v, p_{i+1}(v)) \leq \delta(u, p_i(u)) + \delta(u, v)$ for each even i that satisfies the condition $w \notin B(v)$ in the while loop

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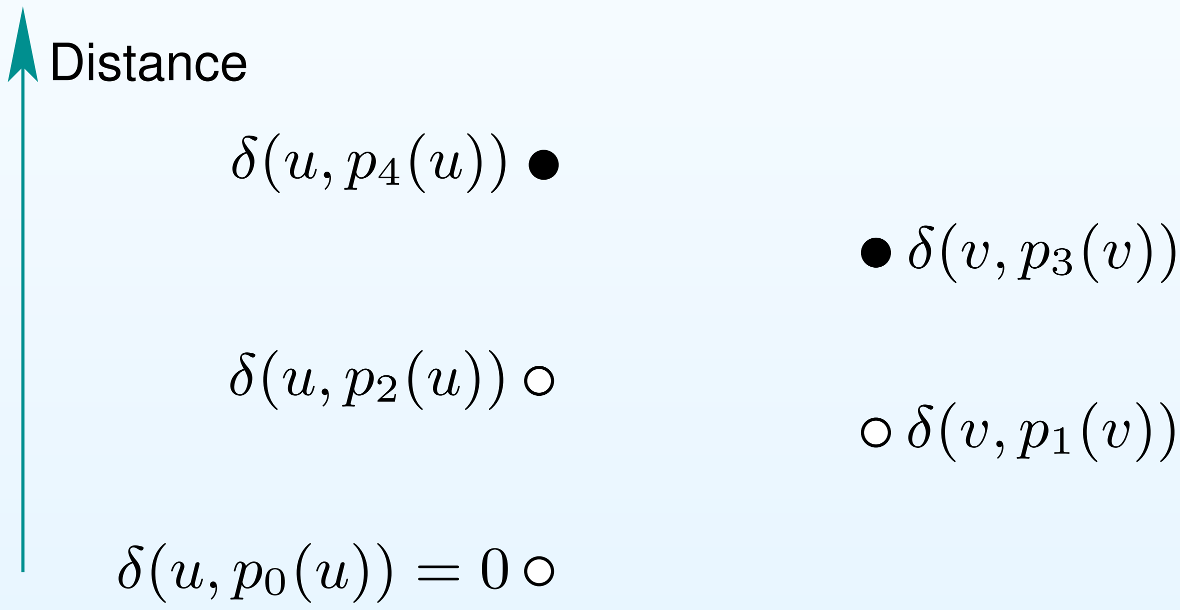

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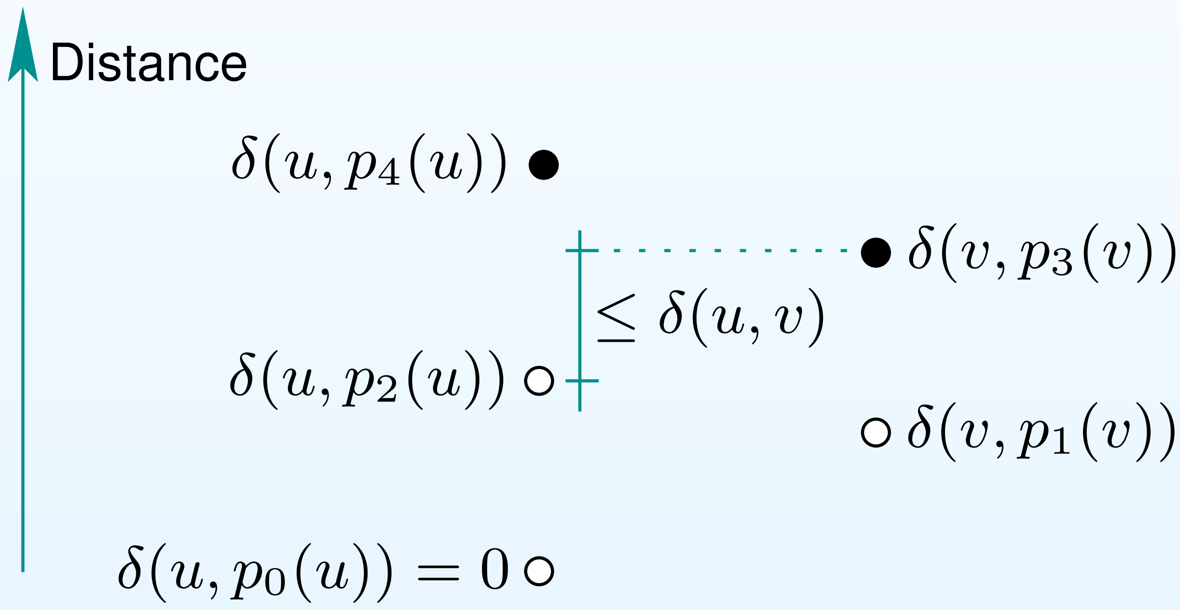

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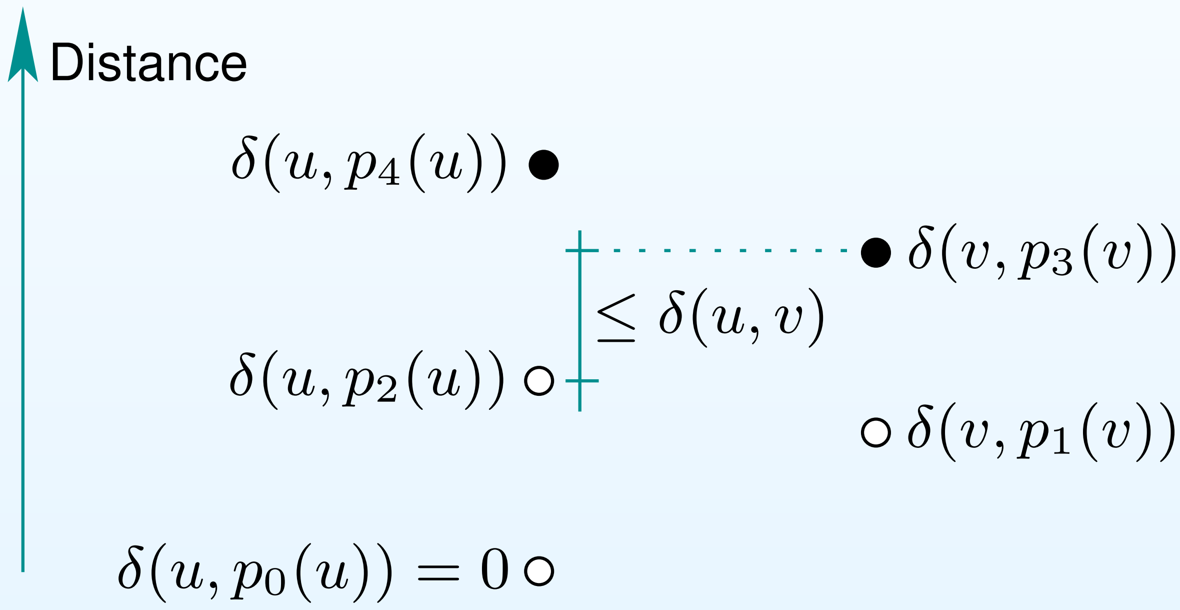

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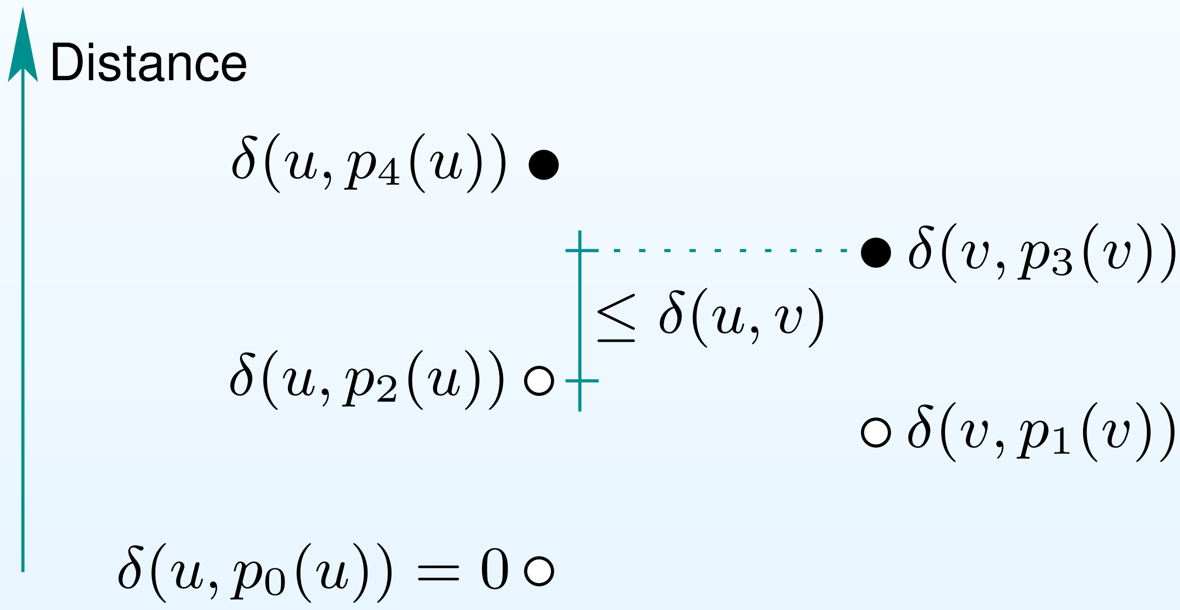

- By a symmetric argument, $\delta(u, p_{i+1}(u)) \leq \delta(v, p_i(v)) + \delta(u, v)$ for each odd i satisfying the condition in the while loop
- Every gap in the figure satisfying the condition is thus $\leq \delta(u, v)$

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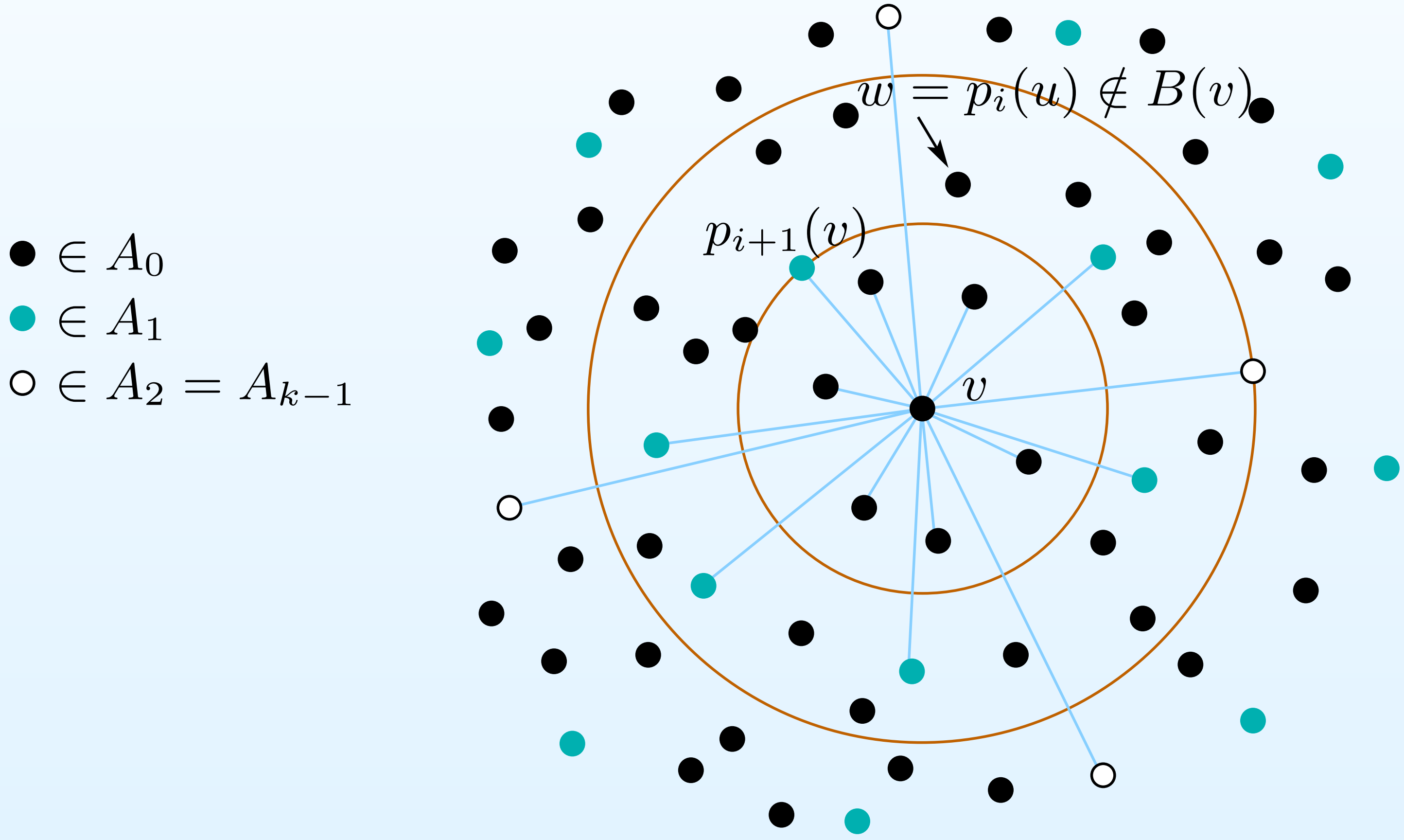

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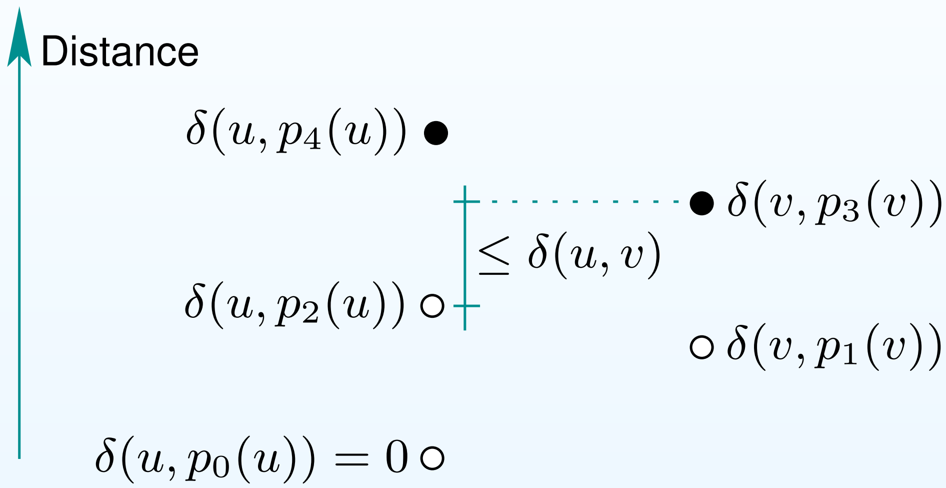
- Combining these inequalities gives the desired:

$$\delta(v, p_{i+1}(v)) \leq \delta(u, p_i(u)) + \delta(u, v)$$

Showing the $(2k-1)$-stretch bound

- We have shown the following for all gaps satisfying the while-loop condition:



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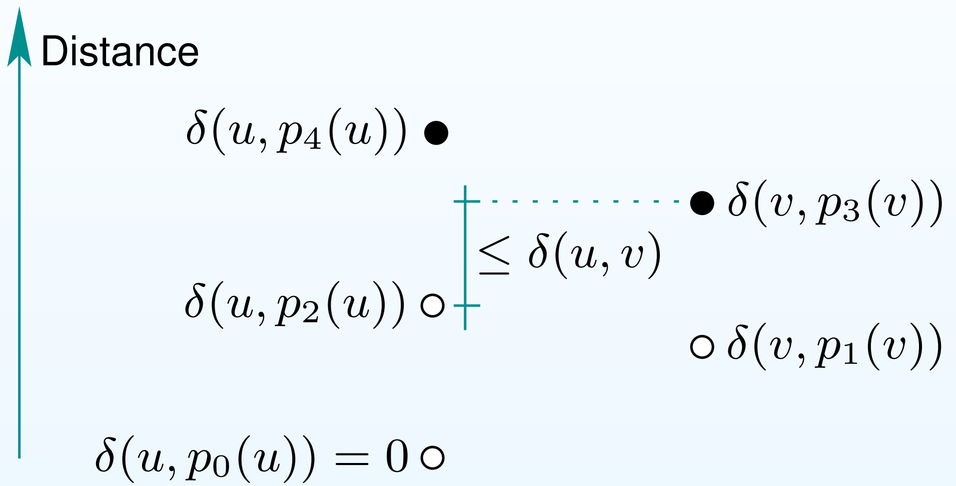

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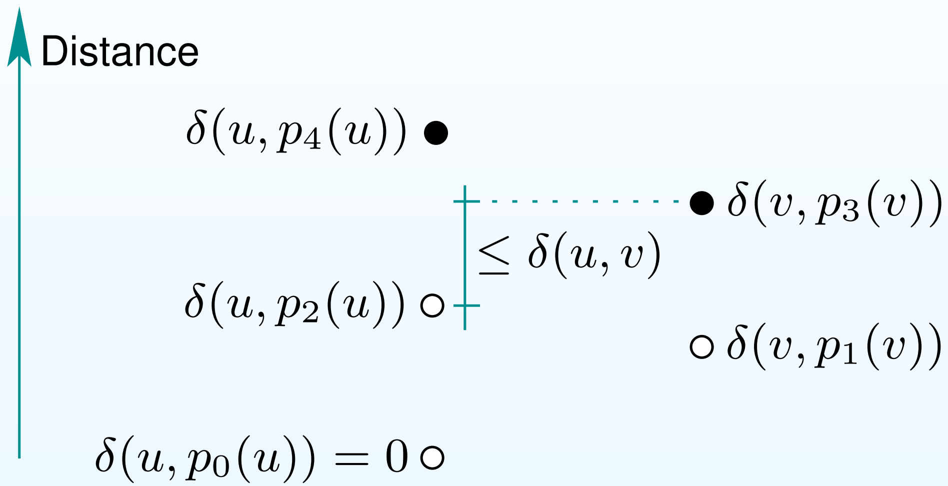

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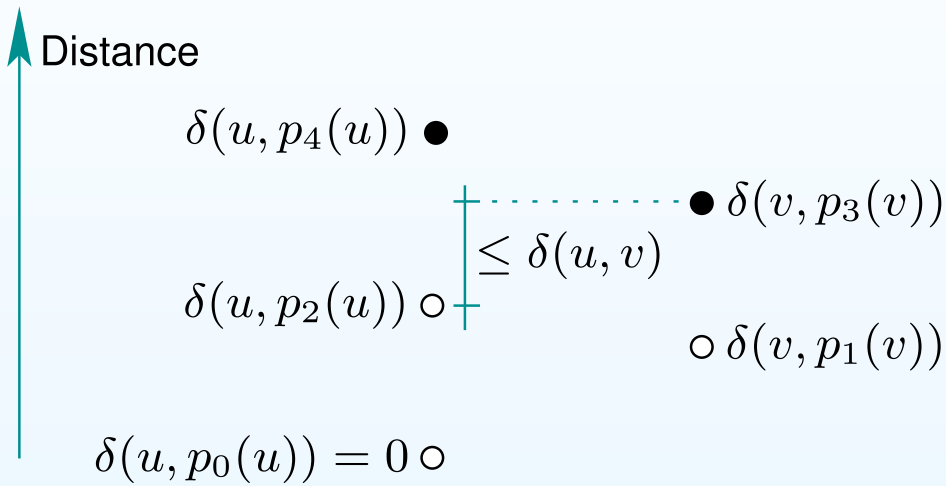

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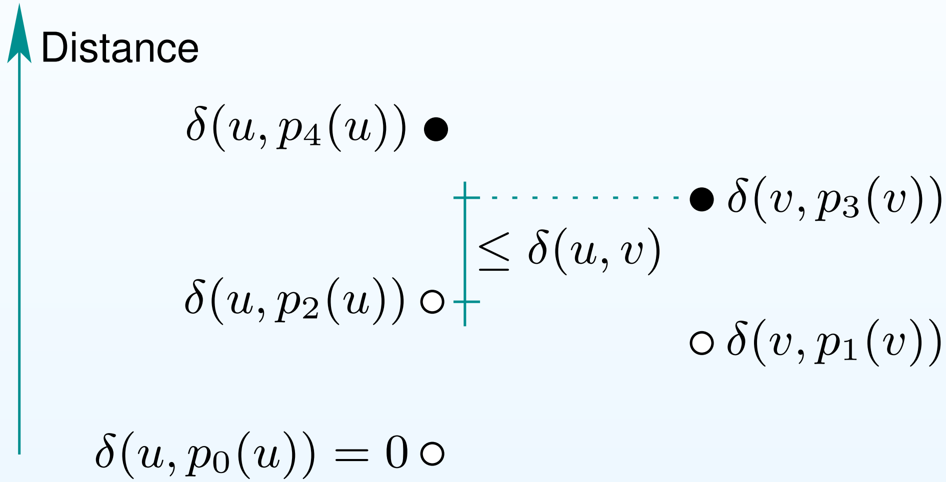

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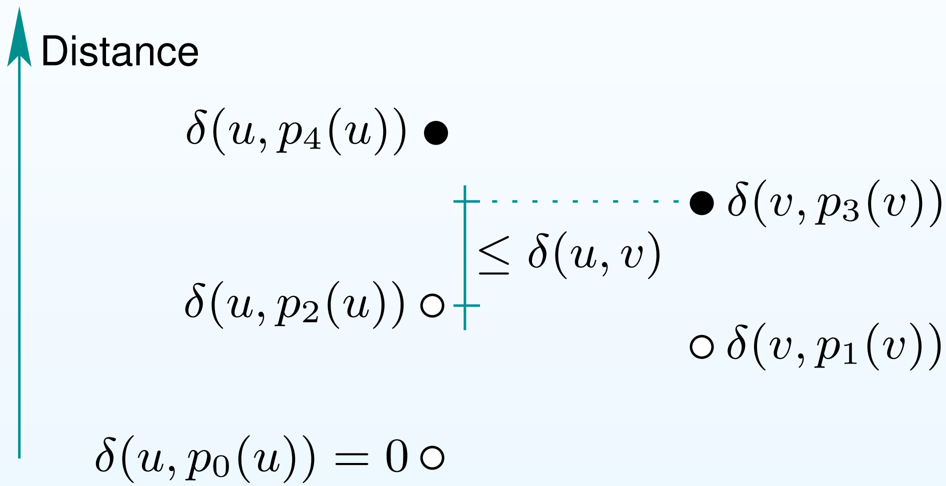

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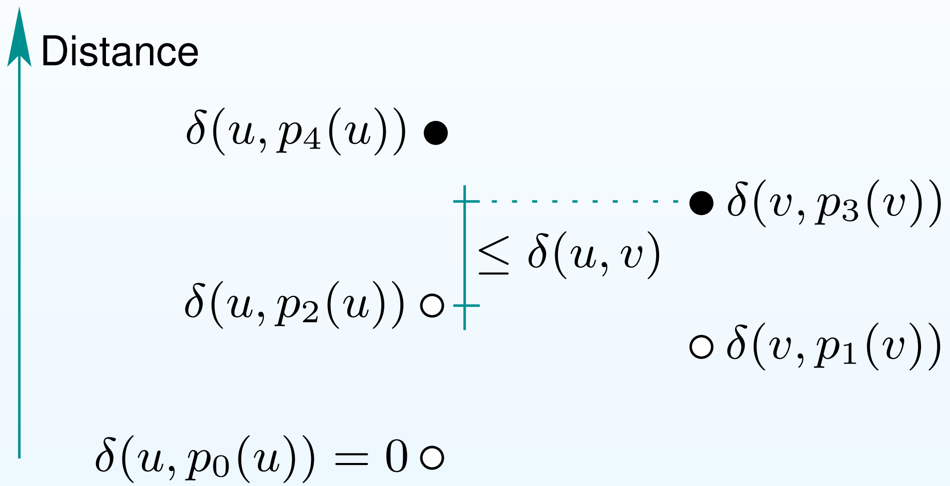

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