Randomized Coloring on a Bounded Degree Graph

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Algorithmic Techniques for Modern Data Models
DTU

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Overview for today

- Randomized distributed algorithm

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  - Monte Carlo and Las Vegas algorithms

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- We focus on Las Vegas algorithms in the following

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- Here, “time” means “number of rounds”

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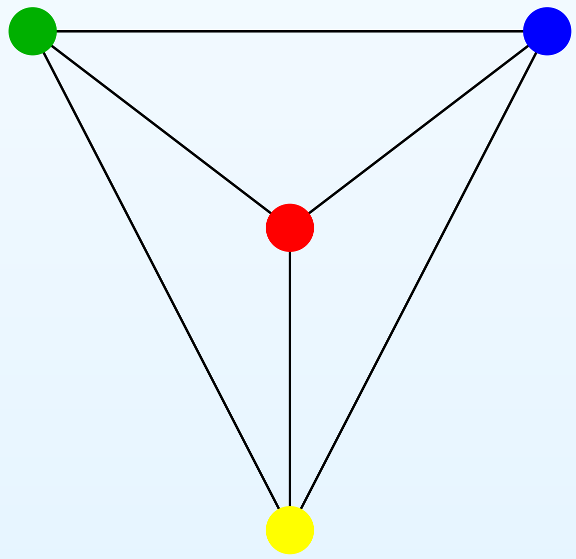

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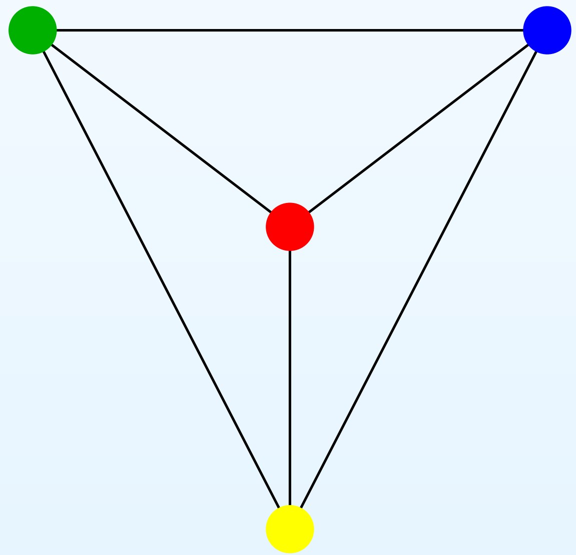

- We will present a Las Vegas distributed algorithm running in $O(\log n)$ rounds w.h.p.

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  - We will show that its time bound is $O(\log n)$ w.h.p.

Refined Algorithm: Information Maintained at Node u

- Define the *color palette* of $u \in V$ as

$$C(u) = \{1, 2, \ldots, \deg_G(u) + 1\}$$

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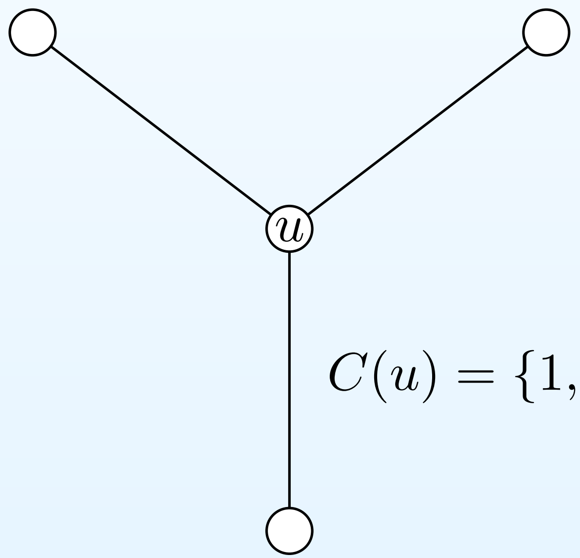

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- At termination, the $c(u)$-colors will form a valid $(\Delta + 1)$-coloring

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- u still sends $c(u)$ to its neighbors after stopping

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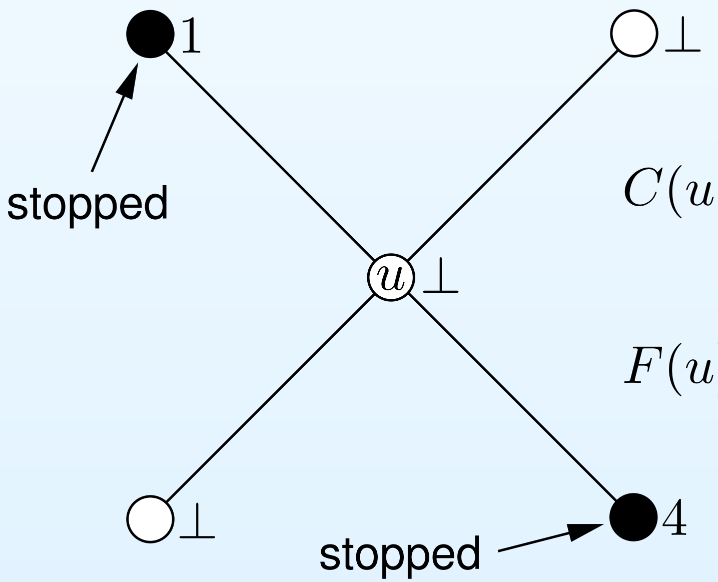

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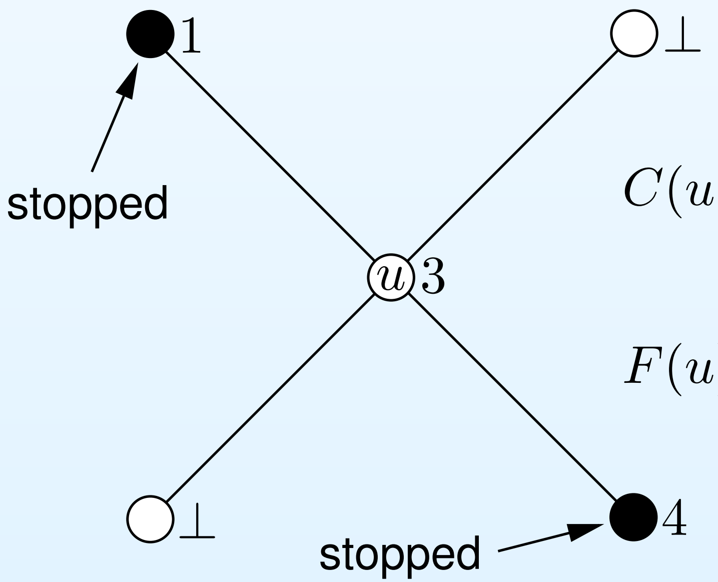

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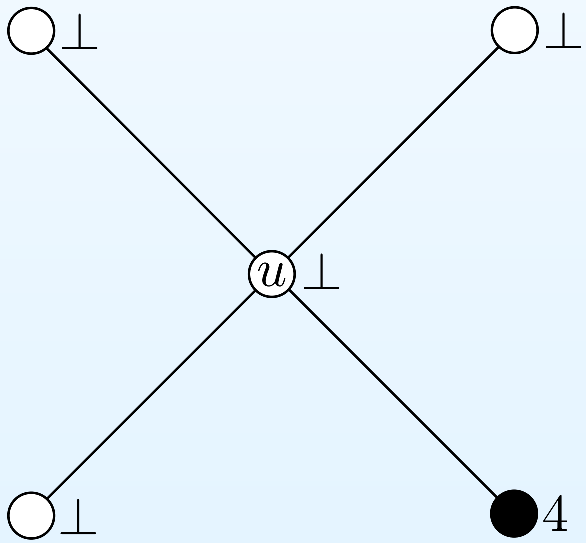

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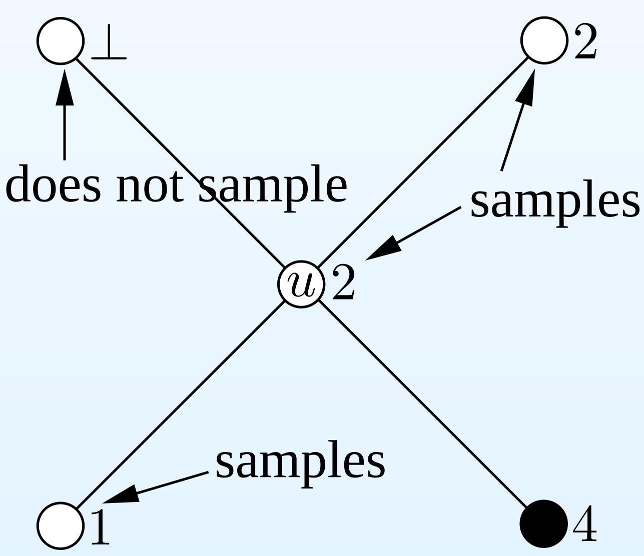

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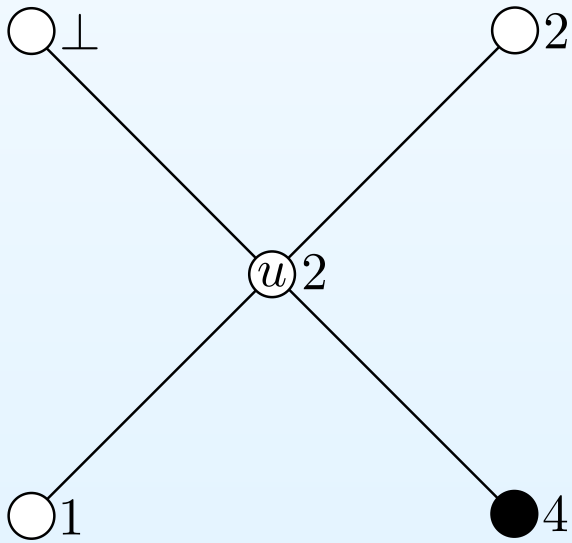

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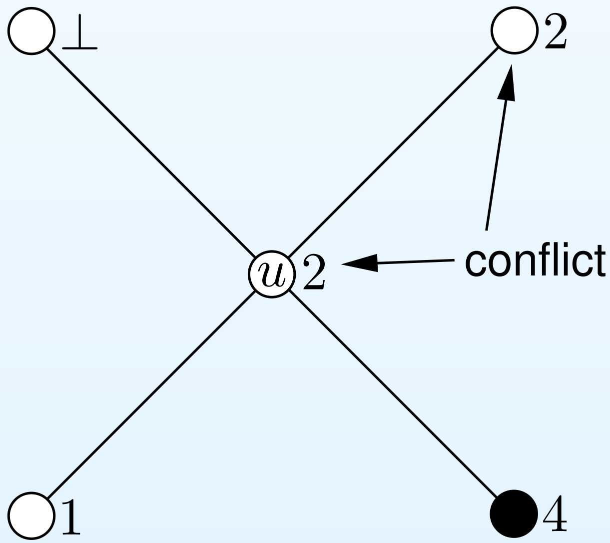

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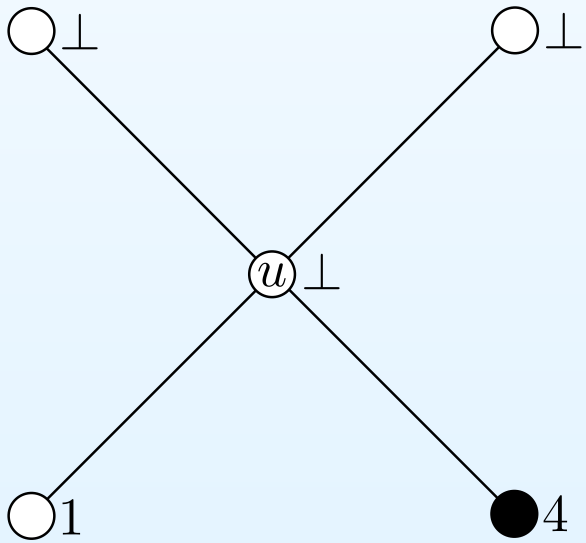

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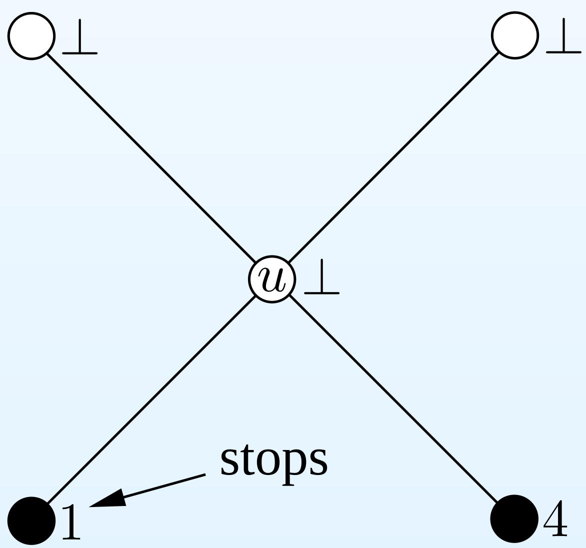

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- Thus, u stops at the end of an even round if:
  - it was assigned a color in the previous (odd) round, and
  - there is no conflict in the current (even) round

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- Since each $c(u) \in C(u) = \{1, \ldots, \deg_G(u) + 1\}$, the output is a $(\Delta + 1)$-coloring

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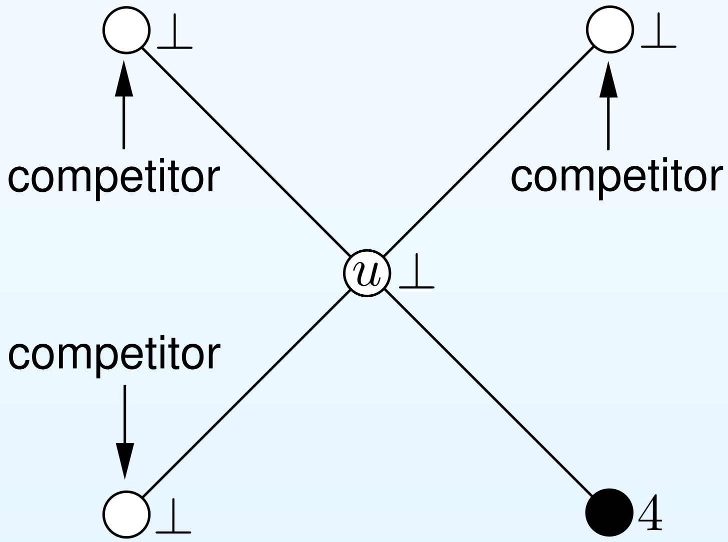

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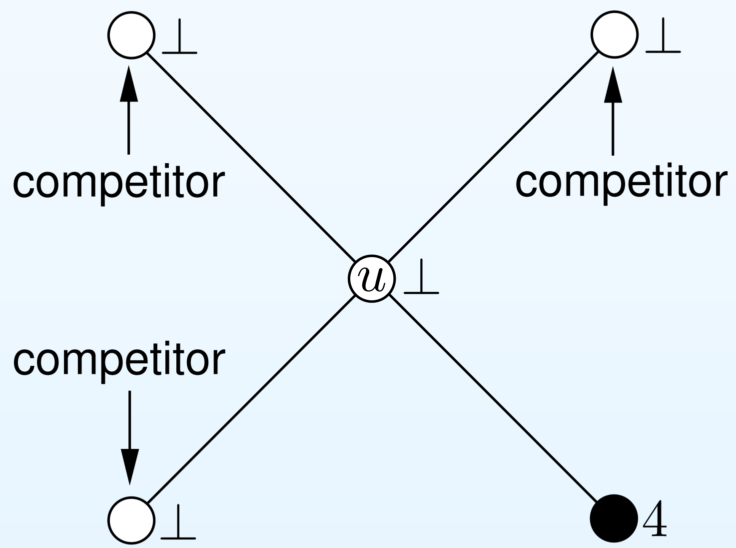

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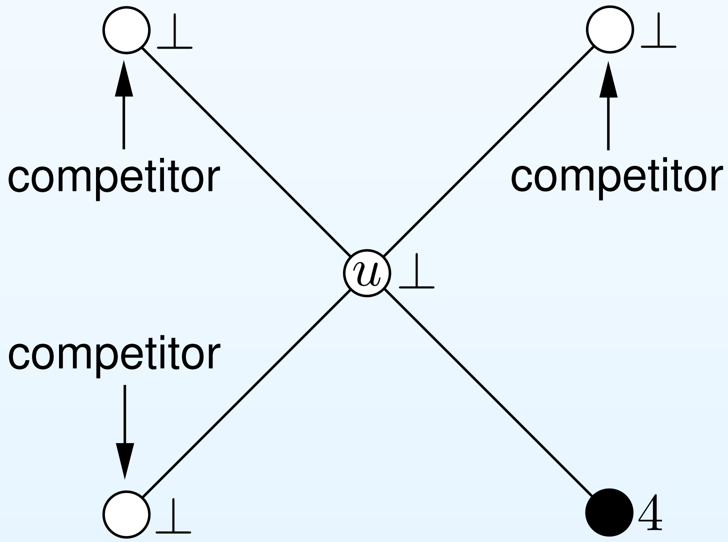

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- We now formally prove that it stops in $O(\log n)$ rounds w.h.p.

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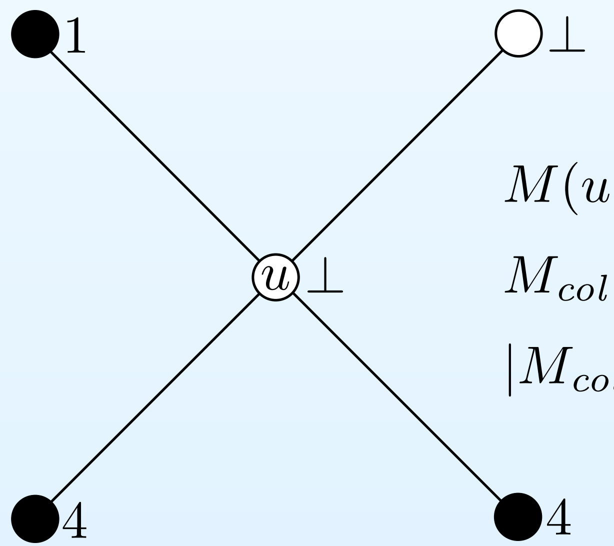

$M(u) = \{1, 4, \perp\}$

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- We pick t so that this is n^{-c} for constant $c > 0$:

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- Have shown: in every *second* round, u stops with probability $> \frac{1}{4}$
- Probability that u has not stopped after $t \in \{2, 4, \ldots\}$ rounds is less than

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- By a union bound over all $u \in V$, the probability that the algorithm has not stopped after t rounds is less than

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- Thus, the algorithm runs in $O(\log n)$ time w.h.p.