02941 Physically Based Rendering

Reflection and Transmission

Jeppe Revall Frisvad

June 2016

Multiscale light modelling

- Light at different scales:
  - Quantum electrodynamics (photons)
  - Electromagnetic radiation (waves)
  - Geometrical optics (rays)
  - Radiative transfer (ray bundles)
- Why do we need to know? To understand:
  - how light interacts with materials;
  - where the material properties come from;
  - when rays don't work.
- Geometrical optics include:
  - the law of reflection (Euclid ∼300 B.C. and before);
  - the law of refraction (Ibn Sahl ∼984, Snel van Royen 1621).
- Electromagnetic radiation (Maxwell 1873) includes:
  - the index of refraction;
  - the amount of reflection and transmission at a specular surface (Fresnel 1832).

Time-harmonic Maxwell equations for isotropic materials

$$\begin{aligned} \nabla \times \boldsymbol{H}_c &= (\sigma - i\omega\varepsilon)\boldsymbol{E}_c \\ \nabla \times \boldsymbol{E}_c &= i\omega\mu\boldsymbol{H}_c \\ \nabla \cdot (\varepsilon \boldsymbol{E}_c) &= 0 \\ \nabla \cdot (\mu \boldsymbol{H}_c) &= 0 \ , \end{aligned}$$

where

- $\omega = 2\pi c/\lambda$ is the angular frequency of the light
- $\mathrm{Re}(\boldsymbol{E}_c)$ is the electric field vector
- $\mathrm{Re}(\boldsymbol{H}_c)$ is the magnetic vector
- and the following are the *isotropic material properties*:
  - σ is the conductivity
  - ε is the permittivity
  - μ is the permeability

The plane wave solution

- Reflection and refraction occurs at a surface marking the boundary between two (at least locally) *homogeneous* materials.
- Plane waves satisfy the time-harmonic Maxwell equations for isotropic and *homogeneous* materials.
- The plane wave equations are:

$$\begin{aligned} \boldsymbol{E}_c(\boldsymbol{x}, t) &= \boldsymbol{E}_0 e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \\ \boldsymbol{H}_c(\boldsymbol{x}, t) &= \boldsymbol{H}_0 e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \ . \end{aligned}$$

where

- $\boldsymbol{x}$ is a position in space
- t is time
- $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are wave amplitudes
- $\boldsymbol{k}$ is the wave vector.

Plane wave Maxwell equations

- The plane wave solution inserted in the time-harmonic Maxwell equations for isotropic, homogeneous materials:

$$\begin{aligned}\boldsymbol{k} \times \boldsymbol{H}_0 &= -\omega(\varepsilon + i\sigma/\omega)\boldsymbol{E}_0 \\ \boldsymbol{k} \times \boldsymbol{E}_0 &= \omega\mu\boldsymbol{H}_0 \\ \boldsymbol{k} \cdot \boldsymbol{E}_0 &= 0 \\ \boldsymbol{k} \cdot \boldsymbol{H}_0 &= 0 \ .\end{aligned}$$

- This boils down to the following conditions that plane waves must satisfy to be physically realisable:

$$\boldsymbol{k} \cdot \boldsymbol{E}_0 = \boldsymbol{k} \cdot \boldsymbol{H}_0 = \boldsymbol{E}_0 \cdot \boldsymbol{H}_0 = 0$$
$$\boldsymbol{k} \cdot \boldsymbol{k} = \omega^2\mu(\varepsilon + i\sigma/\omega) \ .$$

- Thus $\boldsymbol{k}$ is a complex vector if the material is a conductor.

The meaning of the wave vector

- Consider the exponential part of the plane wave equation:

$$e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} = e^{-i\omega t} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

- The first part concerns temporal aspects, the second part concerns spatial aspects.
- The wave vector $\boldsymbol{k}$ determines the spatial aspect (the wave propagation).
- If the material is a conductor, $\boldsymbol{k} = \boldsymbol{k}' + i\boldsymbol{k}''$ is complex:

$$e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} = e^{i\,\boldsymbol{k}'\cdot\boldsymbol{x}} e^{-\boldsymbol{k}''\cdot\boldsymbol{x}} \ .$$

- This reveals that the phase velocity is $v = \omega/k'$ and
  - $\boldsymbol{k}'$ is normal to the surface of constant phase
  - $\boldsymbol{k}''$ is normal to the surface of constant amplitude
- They determine the direction and damping of the wave.

The index of refraction (IOR)

- Recall the only remaining condition which involves material properties:

$$\boldsymbol{k} \cdot \boldsymbol{k} = \omega^2\mu(\varepsilon + i\sigma/\omega) \ .$$

- This is a golden opportunity to reduce the number of material properties.
- Suppose we introduce

$$n = n' + in'' = c\sqrt{\mu(\varepsilon + i\sigma/\omega)} \ .$$

- Then the condition becomes

$$\boldsymbol{k} \cdot \boldsymbol{k} = \frac{\omega^2}{c^2} n^2 \ ,$$

  where
  - c is the speed of light *in vacuo*.
  - n is called the (complex) index of refraction.

Phase velocity and amplitude damping

- A closer look at the condition which describes the relation between material and wave propagation:

$$\boldsymbol{k} \cdot \boldsymbol{k} = \boldsymbol{k}' \cdot \boldsymbol{k}' - \boldsymbol{k}'' \cdot \boldsymbol{k}'' + i2\boldsymbol{k}' \cdot \boldsymbol{k}'' = \frac{\omega^2}{c^2} n^2 \ .$$

- For materials that are not strong absorbers: $\boldsymbol{k}'' \cdot \boldsymbol{k}'' \approx 0$.
- Then (equating the real parts):

$$k' \approx \frac{\omega}{c} n' \quad \Rightarrow \quad v = \frac{\omega}{k'} \approx \frac{c}{n'}$$

  the phase velocity v is determined by the real part of the IOR.
- And (equating the imaginary parts):

$$2k'k'' \cos\theta = \frac{\omega^2}{c^2} 2n'n'' \quad \Rightarrow \quad k'' \approx \frac{\omega}{c} \frac{n''}{\cos\theta} \approx \frac{\omega}{c} n'' \ ,$$

  where θ (usually close to 0) is the angle between $\boldsymbol{k}'$ and $\boldsymbol{k}''$.

The Poynting vector and absorption

- The Poynting vector $\boldsymbol{S}$ determines the direction and magnitude of energy propagation in an electromagnetic field.
- For the plane wave solution, we have:

$$\boldsymbol{S} = \varepsilon_0 c^2 \mu \,\mathrm{Re}(\boldsymbol{E}_c) \times \mathrm{Re}(\boldsymbol{H}_c) \ .$$

- Inserting the plane wave equations and taking the magnitude of $\boldsymbol{S}$ determines the absorption of energy by the material:

$$|\boldsymbol{S}| = \varepsilon_0 c^2 \mu \left| \mathrm{Re}(\boldsymbol{E}_0 e^{-i(\omega t - \boldsymbol{k}' \cdot \boldsymbol{x})}) \times \mathrm{Re}(\boldsymbol{H}_0 e^{-i(\omega t - \boldsymbol{k}' \cdot \boldsymbol{x})}) \right| e^{-2\boldsymbol{k}'' \cdot \boldsymbol{x}} \ .$$

- This reveals that energy is exponentially attenuated by $2k''$, or

$$\sigma_a = 2k'' \approx 2\frac{\omega}{c} n'' = \frac{4\pi n''}{\lambda} \ ,$$

where σ_a is called the *absorption coefficient* and λ is the wavelength *in vacuo*.

Plane waves incident on surfaces

- Consider a plane wave incident on a smooth surface.
- The wave gives rise to two other waves. Altogether we have
  - An incident wave (subscript i)
  - A reflected wave (subscript r)
  - A transmitted wave (subscript t)
- We resolve all waves into two independent components:
  - The wave with the electric vector perpendicular to the plane of incidence: $\perp$-polarised light.
  - The wave with the electric vector parallel to the plane of incidence: $\|$-polarised light.
- Maxwell's equations require that the field vectors are continuous across the surface boundary. Let us call this the *continuity condition*.
- The continuity condition must hold at all times and no matter where we place the point of incidence in space.

Conditions imposed by Maxwell's equations

- Using the continuity condition:

$$\boldsymbol{E}_{\perp i} + \boldsymbol{E}_{\perp r} = \boldsymbol{E}_{\perp t} \ .$$

- This must also hold for $\boldsymbol{x} = 0$, therefore

$$\boldsymbol{E}_{0i}^{\perp} e^{-i\omega_i t} + \boldsymbol{E}_{0r}^{\perp} e^{-i\omega_r t} = \boldsymbol{E}_{0t}^{\perp} e^{-i\omega_t t} \ .$$

- This is true only if

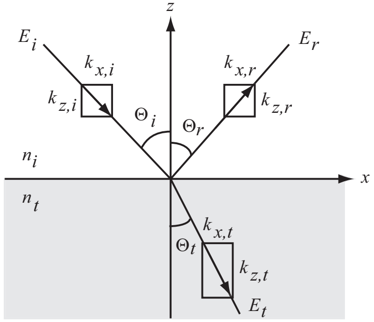
$$\omega_i = \omega_r = \omega_t \ .$$

- With the condition $\boldsymbol{k} \cdot \boldsymbol{k} = n^2\omega^2/c^2$, this reveals

$$\frac{\boldsymbol{k}_i \cdot \boldsymbol{k}_i}{n_i^2} = \frac{\boldsymbol{k}_r \cdot \boldsymbol{k}_r}{n_i^2} = \frac{\boldsymbol{k}_t \cdot \boldsymbol{k}_t}{n_t^2}$$

which shows how the index of refraction governs the propagation of plane waves of light.

The plane of incidence

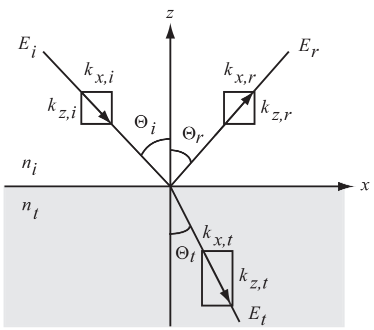


- At the boundary, $z = 0$, we have at the time $t = 0$:

$$\boldsymbol{E}_{0i}^{\perp} e^{i(k_{x,i}x + k_{y,i}y)} + \boldsymbol{E}_{0r}^{\perp} e^{i(k_{x,r}x + k_{y,r}y)} = \boldsymbol{E}_{0t}^{\perp} e^{i(k_{x,t}x + k_{y,t}y)}$$

which must hold for all x and y.

The plane of incidence



- The result is

$$k_{x,i} = k_{x,r} = k_{x,t} \quad \text{and} \quad k_{y,i} = k_{y,r} = k_{y,t} \quad .$$

The plane of incidence

- The result is

$$k_{x,i} = k_{x,r} = k_{x,t} \quad \text{and} \quad k_{y,i} = k_{y,r} = k_{y,t} \quad .$$

- Maxwell's equations require that $\boldsymbol{k}$ is perpendicular to $\boldsymbol{E}_0$.
- Since $\boldsymbol{E}_{0i}^{\perp}$ (by definition) is perpendicular to the plane of incidence, $\boldsymbol{k}_i$ must be parallel to the plane of incidence.
- Then $k_{y,i} = 0$.
- Meaning that $k_{y,r} = k_{y,t} = k_{y,i} = 0$.
- In other words, *the reflected and transmitted vectors lie in the plane of incidence*.

The law of reflection

- Using the following results (found previously):

$$\frac{\boldsymbol{k}_i \cdot \boldsymbol{k}_i}{n_i^2} = \frac{\boldsymbol{k}_r \cdot \boldsymbol{k}_r}{n_i^2}$$

$$k_{x,i} = k_{x,r} \quad \text{and} \quad k_{y,r} = k_{y,i} = 0 \quad .$$

- We find:

$$k_{z,i}^2 = k_{z,r}^2$$

  with the solutions $k_{z,r} = k_{z,i}$ or $k_{z,r} = -k_{z,i}$.
- Only the solution $k_{z,r} = -k_{z,i}$ describes a wave on the right side of the surface.
- The law of reflection is then: *The reflected wave lies in the plane of incidence, the (complex) angle of reflection is equal to the (complex) angle of incidence.*

The law of refraction

- Using the following results (found previously):

$$\frac{\boldsymbol{k}_i \cdot \boldsymbol{k}_i}{n_i^2} = \frac{\boldsymbol{k}_t \cdot \boldsymbol{k}_t}{n_t^2}$$

$$k_{x,i} = k_{x,t}$$

- We find:

$$n_i \frac{k_{x,i}}{\sqrt{\boldsymbol{k}_i \cdot \boldsymbol{k}_i}} = n_t \frac{k_{x,t}}{\sqrt{\boldsymbol{k}_t \cdot \boldsymbol{k}_t}}$$

  or (using complex angles)

$$n_i \sin \Theta_i = n_t \sin \Theta_t \quad .$$

- This is called *generalised Snell's law*.
- The law of refraction is then: *The refracted wave lies in the plane of incidence, the (complex) angle of refraction follows the generalised Snell's law.*

Continuity of the magnetic vector

- One of the plane wave Maxwell equations is:

$$\boldsymbol{H}_0 = \frac{1}{\omega\mu}\boldsymbol{k} \times \boldsymbol{E}_0 \ .$$

- The x component of this equation is

$$H_{0x} = \frac{1}{\omega\mu}(k_y E_{0z} - k_z E_{0y}) \ .$$

- Using the perpendicular components and the result that $k_{y,i} = k_{y,r} = k_{y,t} = 0$, we have

$$H_{0x} = -\frac{k_z}{\omega\mu}E_{0y} \quad \text{and} \quad (0, E_{0y}, 0) = \boldsymbol{E}_0^{\perp} \ .$$

- The continuity condition (invoked on the x component of the magnetic vector) is then

$$-\frac{k_{z,i}}{\omega_i\mu_i}\boldsymbol{E}_{0i}^{\perp} - \frac{k_{z,r}}{\omega_r\mu_i}\boldsymbol{E}_{0r}^{\perp} = -\frac{k_{z,t}}{\omega_t\mu_t}\boldsymbol{E}_{0t}^{\perp} \ .$$

The Fresnel equations for reflection

- The newly found continuity condition:

$$-\frac{k_{z,i}}{\omega_i\mu_i}\boldsymbol{E}_{0i}^{\perp} - \frac{k_{z,r}}{\omega_r\mu_i}\boldsymbol{E}_{0r}^{\perp} = -\frac{k_{z,t}}{\omega_t\mu_t}\boldsymbol{E}_{0t}^{\perp} \ .$$

- Using the law of reflection $k_{z,r} = -k_{z,i}$ and the result that angular frequencies are equal $\omega_i = \omega_r = \omega_t$, we have

$$k_{z,i}\boldsymbol{E}_{0i}^{\perp} - k_{z,i}\boldsymbol{E}_{0r}^{\perp} = k_{z,t}\frac{\mu_i}{\mu_t}\boldsymbol{E}_{0t}^{\perp} \ .$$

- Ignoring magnetic effects, we set $\mu_i/\mu_t = 1$.
- Using the other continuity condition, $\boldsymbol{E}_{0i}^{\perp} + \boldsymbol{E}_{0r}^{\perp} = \boldsymbol{E}_{0t}^{\perp}$, we get

$$k_{z,i}\boldsymbol{E}_{0i}^{\perp} - k_{z,i}\boldsymbol{E}_{0r}^{\perp} = k_{z,t}(\boldsymbol{E}_{0i}^{\perp} + \boldsymbol{E}_{0r}^{\perp}) \quad \Leftrightarrow \quad \boldsymbol{E}_{0r}^{\perp} = \frac{k_{z,i} - k_{z,t}}{k_{z,i} + k_{z,t}}\boldsymbol{E}_{0i}^{\perp} \ .$$

The Fresnel equations for reflection

- The result we arrived at was

$$\boldsymbol{E}_{0r}^{\perp} = \frac{k_{z,i} - k_{z,t}}{k_{z,i} + k_{z,t}}\boldsymbol{E}_{0i}^{\perp} \ .$$

- Diving by $\boldsymbol{E}_{0i}^{\perp}\sqrt{\boldsymbol{k}_i \cdot \boldsymbol{k}_i}$ and using $\cos\Theta = k_z/\sqrt{\boldsymbol{k}_i \cdot \boldsymbol{k}_i}$, we find the amplitude ratio

$$\tilde{r}_{\perp} = \frac{\boldsymbol{E}_{0r}^{\perp}}{\boldsymbol{E}_{0i}^{\perp}} = \frac{n_i\cos\Theta_i - n_t\cos\Theta_t}{n_i\cos\Theta_i + n_t\cos\Theta_t} \ .$$

- In a similar way, we can find

$$\tilde{r}_{\parallel} = \frac{\boldsymbol{E}_{0r}^{\parallel}}{\boldsymbol{E}_{0i}^{\parallel}} = \frac{n_t\cos\Theta_i - n_i\cos\Theta_t}{n_t\cos\Theta_i + n_i\cos\Theta_t} \ .$$

- These are called Fresnel's equations for reflection.

Computing reflectances and transmittances (energy ratios)

- To translate Fresnel's amplitude ratios into energy ratios, one uses the squared absolute value:

$$R_{\perp} = |\tilde{r}_{\perp}|^2 \quad \text{and} \quad R_{\parallel} = |\tilde{r}_{\parallel}|^2 \ .$$

- The Fresnel reflectance for unpolarized light is then

$$R = \frac{1}{2}(R_{\perp} + R_{\parallel}) \ .$$

- The transmittances are one minus the reflectances:

$$T_{\perp} = 1 - R_{\perp} \quad , \quad T_{\parallel} = 1 - R_{\parallel} \quad , \quad T = 1 - R \ .$$

- Please note that the Fresnel equations work with complex indices of refraction.

Reflection and transmission of radiance

- Requiring energy conservation at the surface boundary:

$$\Phi_i = \Phi_r + \Phi_t \ ,$$

  where Φ_i, Φ_r, and Φ_t are the incident, reflected, and transmitted energy fluxes.
- Transformed to radiances:

$$\begin{aligned} L_i \cos\theta_i \, \mathrm{d}A \, \mathrm{d}\omega_i &= L_r \cos\theta_r \, \mathrm{d}A \, \mathrm{d}\omega_r + L_t \cos\theta_t \, \mathrm{d}A \, \mathrm{d}\omega_t \\ &\Updownarrow \\ L_i \cos\theta_i \, \mathrm{d}\omega_i &= L_r \cos\theta_r \, \mathrm{d}\omega_r + L_t \cos\theta_t \, \mathrm{d}\omega_t \ . \end{aligned}$$

- Another way to write the differential solid angles:

$$\begin{aligned} \mathrm{d}\omega_i &= \sin\theta_i \, \mathrm{d}\theta_i \, \mathrm{d}\phi_i \\ \mathrm{d}\omega_r &= \sin\theta_r \, \mathrm{d}\theta_r \, \mathrm{d}\phi_r \\ \mathrm{d}\omega_t &= \sin\theta_t \, \mathrm{d}\theta_t \, \mathrm{d}\phi_t \ . \end{aligned}$$

Using the law of reflection

- Using the law of reflection $\theta_i = \theta_r$ and the plane of incidence $\phi_i = \phi_r = \phi_t$, the solid angles become

$$\begin{aligned} \mathrm{d}\omega_i &= \sin\theta_i \, \mathrm{d}\theta_i \, \mathrm{d}\phi_i \\ \mathrm{d}\omega_r &= \sin\theta_i \, \mathrm{d}\theta_i \, \mathrm{d}\phi_i \\ \mathrm{d}\omega_t &= \sin\theta_t \, \mathrm{d}\theta_t \, \mathrm{d}\phi_i \ . \end{aligned}$$

- By insertion, we can then change the energy conservation condition as follows:

$$\begin{aligned} L_i \cos\theta_i \, \mathrm{d}\omega_i &= L_r \cos\theta_r \, \mathrm{d}\omega_r + L_t \cos\theta_t \, \mathrm{d}\omega_t \\ &\Updownarrow \\ L_i \cos\theta_i \sin\theta_i \, \mathrm{d}\theta_i &= L_r \cos\theta_i \sin\theta_i \, \mathrm{d}\theta_i + L_t \cos\theta_t \sin\theta_t \, \mathrm{d}\theta_t \ . \end{aligned}$$

Compression of solid angle upon transmission

- Using the law of refraction (Snell's law), we have (neglecting the imaginary part of the IOR)

$$\sin\theta_t = \frac{n'_i}{n'_t}\sin\theta_i \quad \text{and} \quad \frac{\mathrm{d}\theta_t}{\mathrm{d}\theta_i} = \frac{\mathrm{d}}{\mathrm{d}\theta_i}\sin^{-1}\left(\frac{n'_i}{n'_t}\sin\theta_i\right) = \frac{n'_i\cos\theta_i}{n'_t\cos\theta_t} \ .$$

- By insertion, we can then change the energy conservation condition as follows:

$$\begin{aligned} L_i \cos\theta_i \sin\theta_i \, \mathrm{d}\theta_i &= L_r \cos\theta_i \sin\theta_i \, \mathrm{d}\theta_i + L_t \cos\theta_t \sin\theta_t \, \mathrm{d}\theta_t \\ &\Updownarrow \\ L_i &= L_r + \left(\frac{n'_i}{n'_t}\right)^2 L_t \ . \end{aligned}$$

- Finally, using Fresnel reflectance R, we have

$$L_o = RL_i + \left(\frac{n'_i}{n'_t}\right)^2 (1-R)L_i \ .$$

02576 Physically Based Rendering

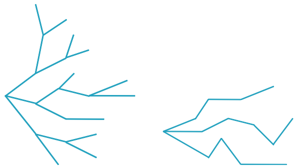
Russian Roulette

Jeppe Revall Frisvad

June 2016

Doing efficient rendering

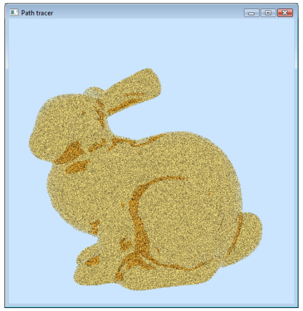
- Time vs. variance.
  - Efficiency = (variance $\times$ rendering time)$^{-1}$.
- Tree vs. path.



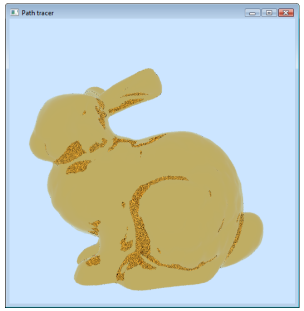
- Splitting vs. Russian roulette.
  - Splitting is a sum over all the branches.
  - Russian roulette is sampling a path.

Demo: The golden bunny

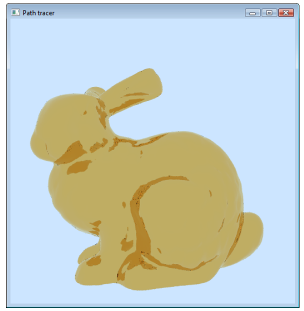
- Using 10 samples per pixel.



Russian roulette
No splitting
Time: 14.4 s



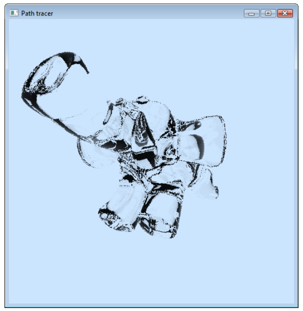
Russian roulette
Split on 1 reflection
Time: 14.8 s



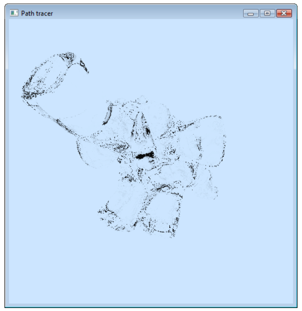
Russian roulette
Split on 2 reflections
Time 14.7 s

Demo: The glass elephant

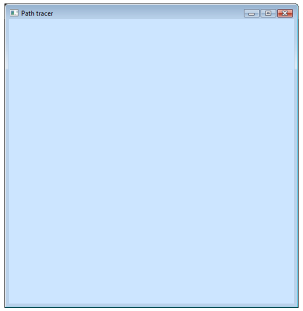
- What does a glass elephant on a flatly coloured background look like?
- Using one sample per pixel.



No Russian roulette
Split on 5 bounces
Time: 4.4 s



No Russian roulette
Split on 10 bounces
Time: 13.7 s



Russian roulette
No splitting
Time 2.6 s

Splitting

- Sum all the events.
  - Example: Reflection multiplied by reflectance.
  - Example: Sum reflection and transmission.
- Exact but expensive.
- When does the recursion stop?
  - Often not before the combinatorial explosion.
  - It might never.
- When to use splitting?
  - Where variance is most striking.
  - Example: The first (or first few) light bounces.

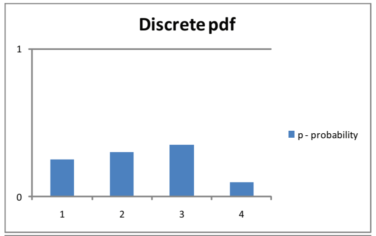
Russian roulette

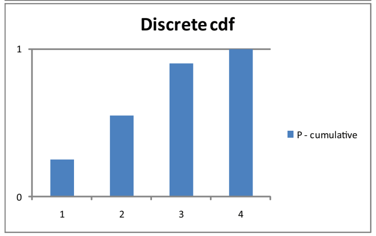
- Either - or.
- Sample an event.
  - Example: Reflection or absorption.
  - Example: Reflection or transmission.
- How? Sample a step function.
- What are the steps?
  - The probabilities of an event.
  - Example: Fresnel reflectance.

Sampling a step function

Algorithm:

sample $\xi \in [0, 1]$ uniformly;
if $(\xi < P_1)$
    call event 1;
    divide by p_1;
else if $(\xi < P_2)$
    call event 2;
    divide by p_2;
else if $(\xi < P_3)$
    ...
else if $(\xi < P_4)$
    ...





Exercises

- Implement recursive ray tracing (reflection and transmission).
- Use Fresnel reflectance with transparent objects.
- MPML (Material Properties Markup Language)
  - The framework matches OBJ material names to MPML material names.
  - If spectral material properties are available in the file `models/media.mpml`, these will be used for OBJ materials of the same name.
- Implement Fresnel with complex IOR.
- Do a metal bunny.
- Do the glass elephant.
- Put them in the Cornell box to make it more interesting.