02941 Physically Based Rendering

Subsurface Scattering

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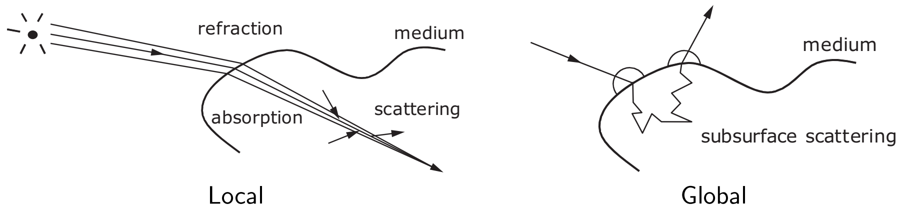
Volumes to surfaces

- From local to global formulation [Preisendorfer 1965].



- Suppose we let $L^0 = T_r(0,s)L(0)$ denote direct transmission.
- Preisendorfer introduces a scattering operator $\boldsymbol{S}_j$ denoting the illumination that has scattered j times inside the medium.
- The scattering operator provides a path from the radiative transfer equation (local) to the rendering equation (global):

$$L = L^0 + \sum_{j=1}^{\infty} \boldsymbol{S}_j L^0 = \sum_{j=0}^{\infty} \boldsymbol{S}_j L^0 \ .$$

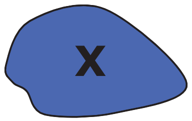
References

- Preisendorfer, R. W. *Radiative Transfer on Discrete Spaces*. Pergamon Press, 1965.

BSSRDF

- Since a monotone, bounded sequence of nonnegative real numbers ($\sum_{j=0}^{n} \boldsymbol{S}_j L^0$) converges to a real number (L), a global solution exists. Thus

$$L = \sum_{j=0}^{\infty} \boldsymbol{S}_j L^0 = \boldsymbol{S} L^0 \ .$$

- But what is $\boldsymbol{S}$?
- For 

$\rightarrow \bullet \boldsymbol{x} \quad , \quad \boldsymbol{S} \rightarrow \sigma_s p(\boldsymbol{x}, \vec{\omega}_i, \vec{\omega}_o) \ .$
- Continuous boundary and interior leads from $\boldsymbol{S}$ to $S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o)$, the so-called Bidirectional Scattering-Surface Reflectance Distribution Function.
- Originally called "the scattering function" [Venable and Hsia 1974] where it included time dependency and inelastic scattering. (Arguably a better name.)

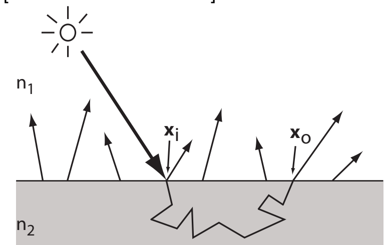
References

- Venable, Jr., W. H., and Hsia, J. J. Optical Radiation Measurement: Describing Spectrophotometric Measurements. Technical report, National Bureau of Standards (US), 1974.

Subsurface scattering

- Behind the rendering equation [Nicodemus et al. 1977]:

$$\frac{\mathrm{d}L_r(\boldsymbol{x}_o, \vec{\omega}_o)}{\mathrm{d}\Phi_i(\boldsymbol{x}_i, \vec{\omega}_i)} = S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o)$$



- An element of reflected radiance $\mathrm{d}L_r$ comes from an element of incident flux $\mathrm{d}\Phi_i$.
- S (the BSSRDF) is the proportionality factor between the two.
- Using the definition of radiance $L = \dfrac{\mathrm{d}^2\Phi}{\cos\theta \, \mathrm{d}A \, \mathrm{d}\omega}$, we have

$$L_r(\boldsymbol{x}_o, \vec{\omega}_o) = \int_A \int_{2\pi} S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) L_i(\boldsymbol{x}_i, \vec{\omega}_i) \cos\theta \, \mathrm{d}\omega_i \, \mathrm{d}A \ .$$

References

- Nicodemus, F. E., Richmond, J. C., Hsia, J. J., Ginsberg, I. W., and Limperis, T. Geometrical considerations and nomenclature for reflectance. Tech. rep., National Bureau of Standards (US), 1977.

Evaluating the rendering equation for subsurface scattering

- The rendering equation for subsurface scattering:

$$L_o(\boldsymbol{x}_o, \vec{\omega}_o) = L_e(\boldsymbol{x}_o, \vec{\omega}_o) + \int_A \int_{2\pi} S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) L_i(\boldsymbol{x}_i, \vec{\omega}_i) \cos\theta \, \mathrm{d}\omega_i \, \mathrm{d}A \ .$$

- Initialize a frame by storing samples of transmitted light.
- For each sample:
  - Sample a point ($\boldsymbol{x}_i$) on the surface of the scattering material:
    - Sample a random triangle in the mesh ($\mathrm{pdf}(\triangle) = A_\triangle / A$, where $A_\triangle$ is triangle area and A is total surface area. Use binary search with the face area cdf).
    - Sample a random point on the triangle ($\mathrm{pdf}(\boldsymbol{x}_{i,\triangle}) = 1/A_\triangle$, use Barycentric coordinates).
  - Sample incident light L_i by sampling $\vec{\omega}_i$ using a cosine-weighted hemisphere ($\mathrm{pdf}(\vec{\omega}_i) = \cos\theta/\pi$), for example.
  - Use $\vec{\omega}_i$ to find the direction of the transmitted ray and the Fresnel transmittance T_i.
  - Compute the transmitted radiance: $L_t = T_i L_i \cos\theta / \text{probabilities} = T_i L_i \pi A$.

Evaluating the rendering equation for subsurface scattering

- The rendering equation for subsurface scattering:

$$L_o(\boldsymbol{x}_o, \vec{\omega}_o) = L_e(\boldsymbol{x}_o, \vec{\omega}_o) + \int_A \int_{2\pi} S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) L_i(\boldsymbol{x}_i, \vec{\omega}_i) \cos\theta \, \mathrm{d}\omega_i \, \mathrm{d}A \ .$$

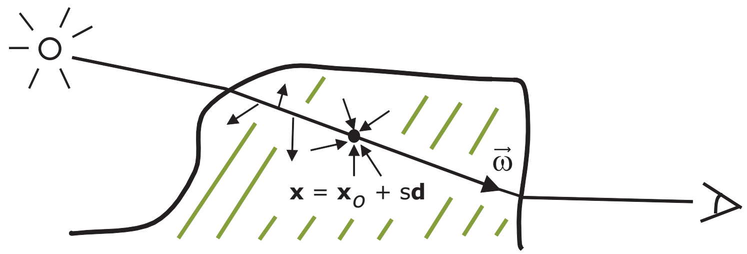
- To shade a ray hitting $\boldsymbol{x}_o$ with direction $-\vec{\omega}_o$:
  - Compute Fresnel transmittance T_o of the ray refracting to $\vec{\omega}_o$ from inside the medium.
  - Accept samples according to a Russian roulette using exponential distance attenuation as probability. [Rejection control.]
  - Use L_t of accepted samples and T_o together with the analytical expression for S to Monte Carlo integrate the rendering equation.
  - The Monte Carlo estimator is:

$$\begin{aligned} L_{d,N,M}(\boldsymbol{x}_o, \vec{\omega}_o) &= \frac{1}{NM} \sum_{p=1}^{M} \sum_{q=1}^{N} \frac{S(\boldsymbol{x}_{i,p}, \vec{\omega}_{i,q}; \boldsymbol{x}_o, \vec{\omega}_o) L_i(\boldsymbol{x}_{i,p}, \vec{\omega}_{i,q}) \cos\theta_i}{\mathrm{pdf}(\boldsymbol{x}_{i,p}) \mathrm{pdf}(\vec{\omega}_{i,q})} \\ &= \frac{1}{NM} \sum_{p=1}^{M} \sum_{q=1}^{N} \frac{T_o S_d(\boldsymbol{x}_{i,p}, \vec{\omega}_{i,q}; \boldsymbol{x}_o, \vec{\omega}_o) L_t(\boldsymbol{x}_{i,p}, \vec{\omega}_{i,q})}{e^{-\sigma_{\mathrm{tr}} \|\boldsymbol{x}_o - \boldsymbol{x}_{i,p}\|}} \left[\xi < e^{-\sigma_{\mathrm{tr}} \|\boldsymbol{x}_o - \boldsymbol{x}_{i,p}\|} \right] . \end{aligned}$$

Radiative transfer as diffusion

- The local formulation: Consider a point $\boldsymbol{x}$ along a ray traversing a medium in the direction $\vec{\omega}$. Then

$$(\vec{\omega} \cdot \nabla) L(\boldsymbol{x}, \vec{\omega}) = -\sigma_t(\boldsymbol{x}) L(\boldsymbol{x}, \vec{\omega}) + \sigma_s(\boldsymbol{x}) \int_{4\pi} p(\boldsymbol{x}, \vec{\omega}', \vec{\omega}) L(\boldsymbol{x}, \vec{\omega}') \, \mathrm{d}\omega' \ .$$

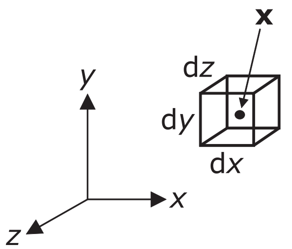


- We assume that the medium is turbid (scattering), but not emissive. (The emission term $L_e(\boldsymbol{x}, \vec{\omega})$ has been left out.)
- To find a global formulation, we think of multiple scattering as a diffusion process.

Fluence and vector irradiance

- In diffusion theory, we use quantities that describe the light field in an element of volume of the scattering medium.
- Total flux, or fluence, is defined by

$$\phi(\boldsymbol{x}) = \int_{4\pi} L(\boldsymbol{x}, \vec{\omega})\, \mathrm{d}\omega \ .$$



- Net flux, or vector irradiance, is

$$\boldsymbol{E}(\boldsymbol{x}) = \int_{4\pi} \vec{\omega} L(\boldsymbol{x}, \vec{\omega})\, \mathrm{d}\omega \ .$$

- $\boldsymbol{E}$ is measured in flux per (orthogonally) projected area.
- The areas are $\mathrm{d}y\,\mathrm{d}z$, $\mathrm{d}x\,\mathrm{d}z$, and $\mathrm{d}x\,\mathrm{d}y$.

Fick's law of diffusion

- The direction in which the fluence undergoes the greatest rate of increase is

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \ .$$

- Intuitively it is then reasonable to assume the proportionality (*Fick's law of diffusion*)

$$\boldsymbol{E}(\boldsymbol{x}) = -D(\boldsymbol{x})\, \nabla\phi(\boldsymbol{x}) \ .$$

- But the assumption does not hold in general(!)
- It is only valid in the asymptotic regions of a medium, that is, in regions far enough from the boundaries to ensure that most light has suffered from multiple scattering events.
- The value of the diffusion coefficient D is also important for the correctness of the law.
- The standard value $D = 1/(3\sigma_t')$ is only valid for nearly isotropic, almost non-absorbing materials.

The diffusion equation

- Integrating the radiative transfer equation over all outgoing directions results in

$$\int_{4\pi} (\vec{\omega} \cdot \nabla) L(\boldsymbol{x}, \vec{\omega})\, \mathrm{d}\omega = -\int_{4\pi} \sigma_t(\boldsymbol{x}) L(\boldsymbol{x}, \vec{\omega})\, \mathrm{d}\omega + \int_{4\pi} \sigma_s(\boldsymbol{x}) \int_{4\pi} p(\boldsymbol{x}, \vec{\omega}', \vec{\omega}) L(\boldsymbol{x}, \vec{\omega}')\, \mathrm{d}\omega'\, \mathrm{d}\omega \, .$$

- In terms of fluence ϕ and vector irradiance $\boldsymbol{E}$, this turns into

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x}) = -\sigma_t(\boldsymbol{x})\phi(\boldsymbol{x}) + \sigma_s(\boldsymbol{x})\phi(\boldsymbol{x}) = -\sigma_a(\boldsymbol{x})\phi(\boldsymbol{x}) \ .$$

- Inserting Fick's law ($\boldsymbol{E} = -D\,\nabla\phi$), we get the diffusion equation (for a non-emitter):

$$\nabla \cdot (D(\boldsymbol{x})\, \nabla\phi(\boldsymbol{x})) = \sigma_a(\boldsymbol{x})\phi(\boldsymbol{x}) \ .$$

  which we can solve for ϕ to get the light field in a scattering medium.

Splitting up the BSSRDF

- Bidirectional scattering surface reflectance distribution function: $S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o)$.
- In asymptotic regions of the medium, we can use diffusion.
- Splitting up the BSSRDF

$$S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) = T_o(\vec{\omega}_o, n_{\mathrm{med}}, n_o) S_d(\|\boldsymbol{x}_o - \boldsymbol{x}_i\|) T_i(\vec{\omega}_i, n_i, n_{\mathrm{med}}) \ + \ \boldsymbol{S}^1 \, .$$

  where
  - T_i and T_o are Fresnel transmittance terms.
  - $\boldsymbol{S}^1$ is the single scattering operator.
  - S_d is the diffusion part.
  - n_{med}, n_i, and n_o are the refractive indices of the scattering medium, the one from where light is incident (incoming), and the one from where light is emergent (outgoing).
- Note that the diffusion part depends only on the distance between the point of incidence ($\boldsymbol{x}_i$) and the point of emergence ($\boldsymbol{x}_o$).

The diffusion part of the BSSRDF

- Consider the diffusion part of the BSSRDF S_d only.

$$\frac{\mathrm{d}L_d(\boldsymbol{x}_o,\vec{\omega}_o)}{\mathrm{d}\Phi_i(\boldsymbol{x}_i,\vec{\omega}_i)} = S_d(\|\boldsymbol{x}_o - \boldsymbol{x}_i\|) \ .$$

- Radiance is $L = \dfrac{\mathrm{d}^2\Phi}{\cos\theta\, dA\, \mathrm{d}\omega}$, radiant exitance is $M = \dfrac{d\Phi}{dA}$.
- By cosine-weighted integration over all outgoing directions, we have

$$\int_{2\pi} \frac{\mathrm{d}L_d(\boldsymbol{x}_o,\vec{\omega}_o)}{\mathrm{d}\Phi_i(\boldsymbol{x}_i,\vec{\omega}_i)} \cos\theta\,\mathrm{d}\omega = \int_{2\pi} S_d(\|\boldsymbol{x}_o - \boldsymbol{x}_i\|)\cos\theta\,\mathrm{d}\omega \ .$$

- Since S_d only depends on distance, we get

$$\frac{\mathrm{d}M_d(\boldsymbol{x}_o)}{\mathrm{d}\Phi_i(\boldsymbol{x}_i,\vec{\omega}_i)} = \pi S_d(\|\boldsymbol{x}_o - \boldsymbol{x}_i\|) \ .$$

Boundary conditions

- Assume that no diffuse radiance is scattered in the inward direction at the surface. Then

$$\left.\begin{aligned} \int_{-2\pi} L(\boldsymbol{x},\vec{\omega}')\cos\theta\,\mathrm{d}\omega' &= 0 \\ \int_{2\pi} L(\boldsymbol{x},\vec{\omega}')\cos\theta\,\mathrm{d}\omega' &= M_d(\boldsymbol{x}) \end{aligned}\right\} \Rightarrow M_d(\boldsymbol{x}) = \int_{4\pi} L(\boldsymbol{x},\vec{\omega}')\cos\theta\,\mathrm{d}\omega' = \vec{n}\cdot\boldsymbol{E}(\boldsymbol{x}).$$

- The assumption rarely holds, but this gives a link between surface and volume rendering (the local and the global formulation of radiative transfer):

$$M_d(\boldsymbol{x}) = \vec{n}\cdot\boldsymbol{E}(\boldsymbol{x}) \ ,$$

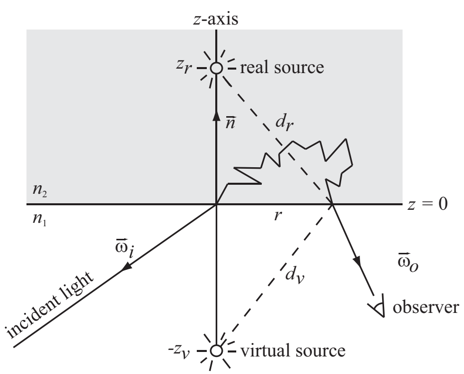
  where $\vec{n}$ is the unit length surface normal.
- This link is crucial for the model. A correction is used later to alleviate the error introduced by the incorrect assumption.

The dipole approximation

- An approximate solution for the diffusion equation:

$$\phi(z) = \frac{\Phi}{4\pi D}\left(\frac{e^{-\sigma_{\mathrm{tr}} d_r}}{d_r} - \frac{e^{-\sigma_{\mathrm{tr}} d_v}}{d_v}\right), \quad \sigma_{\mathrm{tr}} = \sqrt{3\sigma_a\sigma_t'}\,,\ \sigma_t' = \sigma_a + (1-g)\sigma_s\,.$$

- σ_s is the scattering coefficient.
- σ_a is the absorption coefficient.
- g is the asymmetry parameter.



- r is the distance between $\boldsymbol{x}_i$ and $\boldsymbol{x}_o$.

Putting the math together

- Ficks law ($\boldsymbol{E} = -D\,\nabla\phi$) and the boundary conditions:

$$M_d(\boldsymbol{x}) = \vec{n}\cdot\boldsymbol{E}(\boldsymbol{x}) = -D\,\vec{n}\cdot\nabla\phi(\boldsymbol{x}) \ .$$

- Using the rendering equation $\dfrac{\mathrm{d}M_d}{\mathrm{d}\Phi_i} = \pi S_d$:

$$\pi S_d(\|\boldsymbol{x}_o - \boldsymbol{x}_i\|) = -D\frac{\mathrm{d}(\vec{n}\cdot\nabla\phi(\boldsymbol{x}_o))}{\mathrm{d}\Phi_i(\boldsymbol{x}_i,\vec{\omega}_i)} \ .$$

- Inserting the dipole approximation of ϕ:

$$\pi S_d(r) = -D\frac{\mathrm{d}\Phi}{\mathrm{d}\Phi_i}\frac{\partial}{\partial z}\left[\frac{1}{4\pi D}\left(\frac{e^{-\sigma_{\mathrm{tr}} d_r(z,r)}}{d_r(z,r)} - \frac{e^{-\sigma_{\mathrm{tr}} d_v(z,r)}}{d_v(z,r)}\right)\right] \ .$$

  where

$$d_r(z,r) = \sqrt{r^2 + (z+z_r)^2} \quad \text{and} \quad d_v(z,r) = \sqrt{r^2 + (z-z_v)^2} \ .$$

The subsurface scattering model

- Taking the partial derivative with respect to z and afterwards setting $z = 0$, we get:

$$S_d(r) = \frac{1}{4\pi^2}\frac{\mathrm{d}\Phi}{\mathrm{d}\Phi_i}\left(\frac{z_r(1+\sigma_{\mathrm{tr}}d_r)e^{-\sigma_{\mathrm{tr}}d_r}}{d_r^3} + \frac{z_v(1+\sigma_{\mathrm{tr}}d_v)e^{-\sigma_{\mathrm{tr}}d_v}}{d_v^3}\right).$$

  with $d_r(r) = \sqrt{r^2 + z_r^2}$ and $d_v(r) = \sqrt{r^2 + z_v^2}$.
- It remains now only to estimate Φ, z_r, and z_v.
- The model is based on *reduced* scattering properties:
  - Reduced scattering coefficient: $\sigma_s' = \sigma_s(1-g)$.
  - Reduced extinction coefficient: $\sigma_t' = \sigma_s' + \sigma_a$.
  - Reduced scattering albedo: $\alpha' = \sigma_s'/\sigma_t'$.
  - Transport mean free path: $\Lambda = 1/\sigma_t'$.
- From these we define: $\Phi = \alpha'\Phi_i$ (meaning $\frac{d\Phi}{d\Phi_i} = \alpha'$) and $z_r = 1/\sigma_t'$.
- The displacement of the virtual source z_v is corrected to mitigate the boundary condition error: $z_v = z_r + 4AD$, where D is the diffusion coefficient and A is the reflection (or the Groenhuis) parameter.

Validity

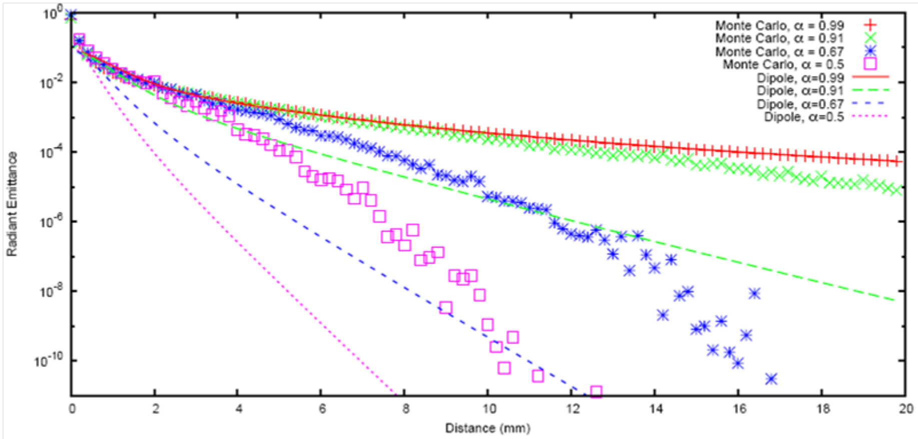


Figure from Craig Donner's PhD thesis [2006].

References

- Donner, C. *Towards Realistic Image Synthesis of Scattering Materials*, PhD thesis, University of California, San Diego, 2006.

Exercises

- Work with the subsurface scattering model.
- You already have single scattering if you did the volume rendering exercises.
- To do diffusion:
  - Sample a position ($\boldsymbol{x}_i$) on the triangle mesh which defines the surface of the scattering material.
  - Sample incident light using the cosine-weighted hemisphere.
  - Compute the diffusion through the material (from $\boldsymbol{x}_i$ to $\boldsymbol{x}_o$) using the dipole approximation.
  - Multiply by the correct Fresnel transmittances.