Directional Subsurface Scattering

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Splitting up the BSSRDF

- Bidirectional Scattering-Surface Reflectance Distribution Function: $S = S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o)$.
- Away from sources and boundaries, we can use diffusion.
- Splitting up the BSSRDF

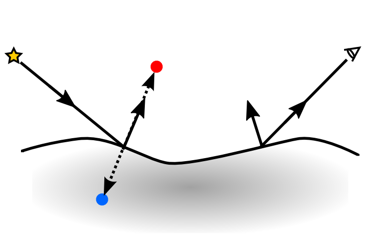
$$S = T_{12}(S^{(0)} + S^{(1)} + S_d)T_{21} .$$

where
  - T_{12} and T_{21} are Fresnel transmittance terms (using $\vec{\omega}_i$, $\vec{\omega}_o$).
  - $S^{(0)}$ is the direct transmission part (using Dirac δ-functions).
  - $S^{(1)}$ is the single scattering part (using all arguments).
  - S_d is the diffusive part (multiple scattering, using $|\boldsymbol{x}_o - \boldsymbol{x}_i|$).
- We distribute the single scattering to the other terms using the delta-Eddington approximation:

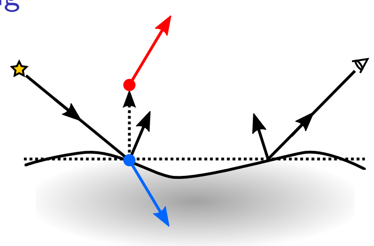
$$S = T_{12}(S_{\delta E} + S_d)T_{21} \ ,$$

and generalize the model such that $S_d = S_d(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o)$.

Analytical models for subsurface scattering



standard dipole



directional dipole

$$S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) = T_{12}(\vec{\omega}_i)(S_1 + S_d(\|\boldsymbol{x}_o - \boldsymbol{x}_i\|))T_{21}(\vec{\omega}_o) .$$

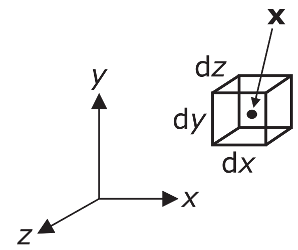
$$S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) = T_{12}(\vec{\omega}_i)(S_{\delta E} + S_d(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o))T_{21}(\vec{\omega}_o) .$$

- Directions ($\vec{\omega}_i$, $\vec{\omega}_o$) also require surface normals ($\vec{n}_i$, $\vec{n}_o$) to get angles (θ_i, θ_o).
- T_{12} and T_{21} are Fresnel transmittances.
- S_1 and $S_{\delta E}$ are fully directional (depend on $\boldsymbol{x}_i$, $\vec{\omega}_i$, $\boldsymbol{x}_o$, $\vec{\omega}_o$, and normals).

Diffusion theory

- Think of multiple scattering as a diffusion process.
- In diffusion theory, we use quantities that describe the light field in an element of volume of the scattering medium.
- Total flux, or fluence, is defined by

$$\phi(\boldsymbol{x}) = \int_{4\pi} L(\boldsymbol{x}, \vec{\omega})\, d\omega \ .$$



- We find an expression for ϕ by solving the diffusion equation

$$(D\nabla^2 - \sigma_a)\phi(\boldsymbol{x}) = -q(\boldsymbol{x}) + 3D\,\nabla\cdot\boldsymbol{Q}(\boldsymbol{x}) \ ,$$

where σ_a and D are absorption and diffusion coefficients, while q and $\boldsymbol{Q}$ are zeroth and first order source terms.

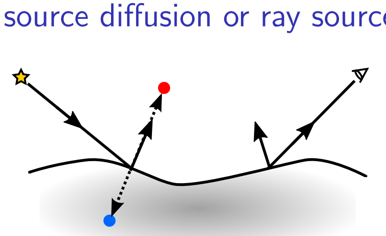
Deriving a BSSRDF

- Assume that emerging light is diffuse due to a large number of scattering events: $S_d(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) = S_d(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o)$.
- Integrating emerging diffuse radiance over outgoing directions, we find

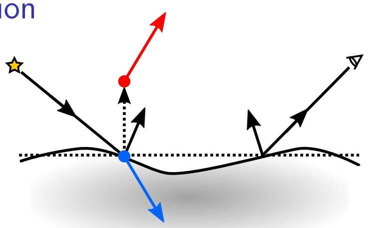
$$S_d = \frac{C_\phi(\eta)\,\phi - C_{\boldsymbol{E}}(\eta)\,D\,\vec{n}_o \cdot \nabla\phi}{\Phi\,4\pi\,C_\phi(1/\eta)} \;,$$

where
  - Φ is the flux entering the medium at $\boldsymbol{x}_i$.
  - $\vec{n}_o$ is the surface normal at the point of emergence $\boldsymbol{x}_o$.
  - C_ϕ and $C_{\boldsymbol{E}}$ depend on the relative index of refraction η and are polynomial fits of different hemispherical integrals of the Fresnel transmittance.
- This connects the BSSRDF and the diffusion theory.
- To get an analytical model, we use a special case solution for the diffusion equation (an expression for ϕ).
- Then, "all" we need to do is to find $\nabla\phi$ (do the math) and deal with boundary conditions (build a plausible model).

Point source diffusion or ray source diffusion



standard dipole



directional dipole

- Point source diffusion [Bothe 1941; 1942]

$$\phi(r) = \frac{\Phi}{4\pi D}\frac{e^{-\sigma_{\text{tr}} r}}{r} \;,$$

where $r = |\boldsymbol{x}_o - \boldsymbol{x}_i|$ and $\sigma_{\text{tr}} = \sqrt{\sigma_a/D}$ is the effective transport coefficient.

- Ray source diffusion [Menon et al. 2005a; 2005b]

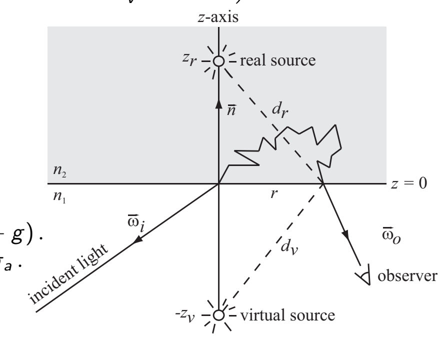
$$\phi(r, \theta) = \frac{\Phi}{4\pi D}\frac{e^{-\sigma_{\text{tr}} r}}{r}\left(1 + 3D\frac{1+\sigma_{\text{tr}} r}{r}\cos\theta\right),$$

where θ is the angle between the refracted ray and $\boldsymbol{x}_o - \boldsymbol{x}_i$.

Diffusive part of the standard dipole

$$S_d(r) = \frac{\alpha'}{4\pi^2}\left(\frac{z_r(1+\sigma_{\text{tr}} d_r)e^{-\sigma_{\text{tr}} d_r}}{d_r^3} + \frac{z_v(1+\sigma_{\text{tr}} d_v)e^{-\sigma_{\text{tr}} d_v}}{d_v^3}\right).$$

- Distances:
  - $z_r = \Lambda$.
  - $z_v = \Lambda + 4AD$.
  - $d_r(r) = \sqrt{r^2 + z_r^2}$.
  - $d_v(r) = \sqrt{r^2 + z_v^2}$.
- Optical properties ($\eta = n_2/n_1$, σ_s, σ_a, g):
  - Reduced scattering coefficient: $\sigma_s' = \sigma_s(1-g)$.
  - Reduced extinction coefficient: $\sigma_t' = \sigma_s' + \sigma_a$.
  - Reduced scattering albedo: $\alpha' = \sigma_s'/\sigma_t'$.
  - Transport mean free path: $\Lambda = 1/\sigma_t'$.
  - Diffusion coefficient: $D = \Lambda/3$.
  - Transport coefficient: $\sigma_{\text{tr}} = \sqrt{\sigma_a/D}$.
  - Reflection parameter: $A(\eta)$ (ratio of polynomial fits).

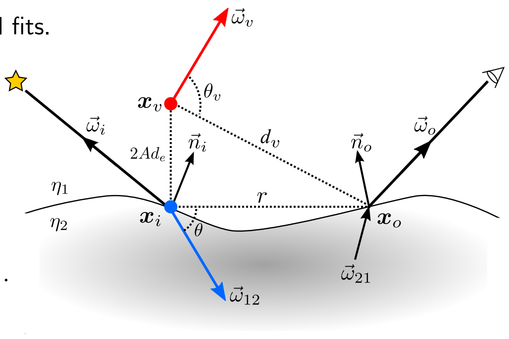


Directional subsurface scattering when disregarding the boundary

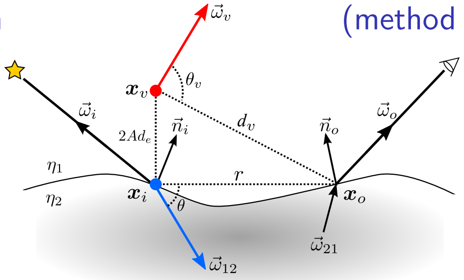
$$S_d'(\boldsymbol{x}, \vec{\omega}_{12}, r) = \frac{1}{4C_\phi(1/\eta)}\frac{1}{4\pi^2}\frac{e^{-\sigma_{\text{tr}} r}}{r^3}\Bigg[C_\phi(\eta)\left(\frac{r^2}{D} + 3(1+\sigma_{\text{tr}} r)\,\boldsymbol{x}\cdot\vec{\omega}_{12}\right)$$
$$- C_{\boldsymbol{E}}(\eta)\left(3D(1+\sigma_{\text{tr}} r)\,\vec{\omega}_{12}\cdot\vec{n}_o - \left((1+\sigma_{\text{tr}} r) + 3D\frac{3(1+\sigma_{\text{tr}} r)+(\sigma_{\text{tr}} r)^2}{r^2}\boldsymbol{x}\cdot\vec{\omega}_{12}\right)\boldsymbol{x}\cdot\vec{n}_o\right)\Bigg],$$

where $C_\phi(\eta)$ and $C_{\boldsymbol{E}}(\eta)$ are polynomial fits.

- Additional dependencies:
  - Normal: $\vec{n}_o$.
  - Optical properites: η, D, σ_{tr}.
- Note the exponential term: $e^{-\sigma_{\text{tr}} r}$.
- Normal incidence: $\vec{\omega}_{12} \cdot \vec{n}_o = \pm 1$.
- Plane (half-space): $\boldsymbol{x} \cdot \vec{n}_o \approx 0$.
- normal incidence on plane: $\boldsymbol{x} \cdot \vec{\omega}_{12} \approx 0$.
- $r \to \|\boldsymbol{x}_o - \boldsymbol{x}_i\|$ for $\|\boldsymbol{x}_o - \boldsymbol{x}_i\| \to \infty$.



Dipole configuration (method of mirror images)



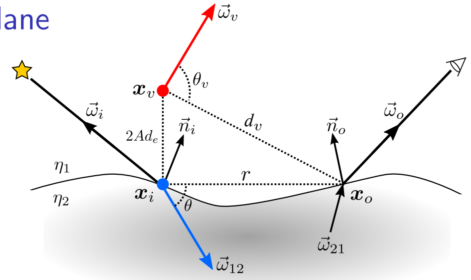
- We place the "real" ray source at the boundary and reflect it in an extrapolated boundary to place the "virtual" ray source.
- Distance to the extrapolated boundary [Davison 1958]:

$$d_e = 2.131\, D/\sqrt{1 - 3D\sigma_a} \ .$$

- In case of a refractive boundary ($\eta_1 \neq \eta_2$), the distance is

$$Ad_e \quad \text{with} \quad A = \frac{1 - C_{\boldsymbol{E}}(\eta)}{2C_\phi(\eta)} \ .$$

Modified tangent plane



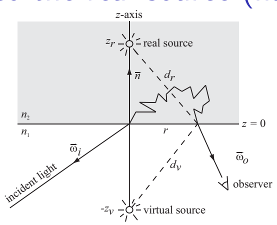
- The dipole assumes a semi-infinite medium.
- We assume that the boundary contains the vector $\boldsymbol{x}_o - \boldsymbol{x}_i$ and that it is perpendicular to the plane spanned by $\vec{n}_i$ and $\boldsymbol{x}_o - \boldsymbol{x}_i$.
- The normal of the assumed boundary plane is then

$$\vec{n}_i^* = \frac{\boldsymbol{x}_o - \boldsymbol{x}_i}{|\boldsymbol{x}_o - \boldsymbol{x}_i|} \times \frac{\vec{n}_i \times (\boldsymbol{x}_o - \boldsymbol{x}_i)}{|\vec{n}_i \times (\boldsymbol{x}_o - \boldsymbol{x}_i)|}, \quad \text{or } \vec{n}_i^* = \vec{n}_i \text{ if } \boldsymbol{x}_o = \boldsymbol{x}_i.$$

and the virtual source is given by

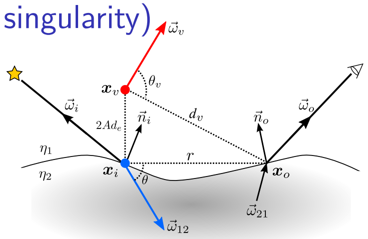
$$\boldsymbol{x}_v = \boldsymbol{x}_i + 2Ad_e\vec{n}_i^* \ , \ \ d_v = |\boldsymbol{x}_v - \boldsymbol{x}_i| \ , \ \ \vec{\omega}_v = \vec{\omega}_{12} - 2(\vec{\omega}_{12} \cdot \vec{n}_i^*)\vec{n}_i^* \ .$$

Distance to the real source (handling the singularity)



standard dipole

$$d_r = \sqrt{r^2 + z_r^2} \ .$$



directional dipole

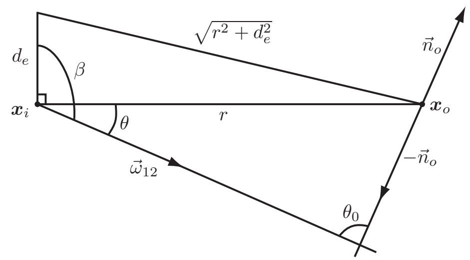
$$d_r = r \ ?$$

- Emergent radiance is an integral over z of a Hankel transform of a Green function which is Fourier transformed in x and y.
- Approximate analytic evaluation is possible if r is corrected to

$$R^2 = r^2 + (z' + d_e)^2 \ .$$

- The resulting model for $z' = 0$ corresponds to the standard dipole where $z' = z_r$ and d_e is replaced by the virtual source.

Distance to the real source (handling the singularity)



- Since we neither have normal incidence nor $\boldsymbol{x}_o$ in the tangent plane, we modify the distance correction:

$$R^2 = r^2 + z'^2 + d_e^2 - 2z'd_e\cos\beta \ .$$

- The integral over z can be reformulated as an integral along the refracted ray.
- We can approximate this integral by choosing an offset D^* along the refracted ray. Then $z' = D^*|\cos\theta_0|$.

Diffusive part of the directional dipole

$$S_d(\mathbf{x}_i, \vec{\omega}_i; \mathbf{x}_o) = S'_d(\mathbf{x}_o - \mathbf{x}_i, \vec{\omega}_{12}, d_r) - S'_d(\mathbf{x}_o - \mathbf{x}_v, \vec{\omega}_v, d_v) \ ,$$

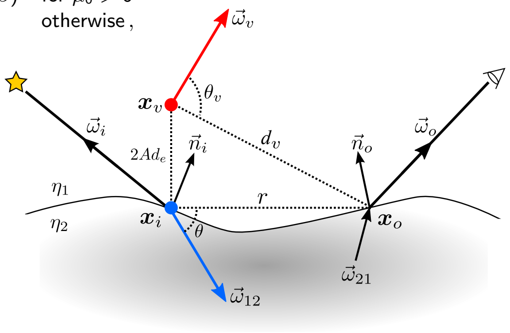
- Real source:
  - $\vec{\omega}_{12} = \eta^{-1}((\vec{\omega}_i \cdot \vec{n}_i)\vec{n}_i - \vec{\omega}_i) - \vec{n}_i\sqrt{1 - \eta^{-2}(1 - (\vec{\omega}_i \cdot \vec{n}_i)^2)}$.
  - $d_r^2 = \begin{cases} \|\mathbf{x}_o - \mathbf{x}_i\|^2 + D\mu_0(D\mu_0 - 2d_e\cos\beta) & \text{for } \mu_0 > 0 \\ \|\mathbf{x}_o - \mathbf{x}_i\|^2 + 1/(3\sigma_t)^2 & \text{otherwise}, \end{cases}$

    with $\mu_0 = \cos\theta_0 = -\vec{n}_o \cdot \vec{\omega}_{12}$

    and $\cos\beta = -\sqrt{\frac{r^2 - (\mathbf{x}\cdot\omega_{12})^2}{r^2 + d_e^2}}$.
- Virtual source:
  - Modified normal:

    $\vec{n}_i^* = \frac{\mathbf{x}_o - \mathbf{x}_i}{\|\mathbf{x}_o - \mathbf{x}_i\|} \times \frac{\vec{n}_i \times (\mathbf{x}_o - \mathbf{x}_i)}{\|\vec{n}_i \times (\mathbf{x}_o - \mathbf{x}_i)\|}$,

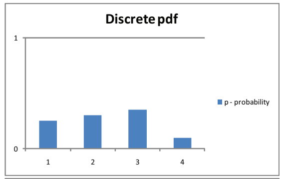
    or $\vec{n}_i^* = \vec{n}_i$ if $\mathbf{x}_o = \mathbf{x}_i$.
  - $\mathbf{x}_v = \mathbf{x}_i + 2Ad_e\vec{n}_i^*$, $d_v = |\mathbf{x}_v - \mathbf{x}_i|$, $\vec{\omega}_v = \vec{\omega}_{12} - 2(\vec{\omega}_{12} \cdot \vec{n}_i^*)\vec{n}_i^*$.

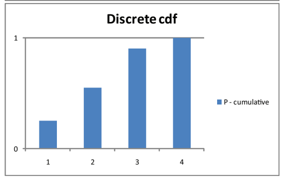


Rejection control

- The exponential attenuation $e^{-\sigma_{tr}d}$ appears in all analytical BSSRDFs and $d \to \|\mathbf{x}_o - \mathbf{x}_i\|$ for $\|\mathbf{x}_o - \mathbf{x}_i\| \to \infty$.
- We should exploit this.
- Russian roulette:

  sample $\xi \in [0, 1]$ uniformly;
  if ($\xi < P_1$)
  call event 1;
  divide by p_1;
  else if ($\xi < P_2$)
  call event 2;
  divide by p_2;
  else if ($\xi < P_3$)
  ...
  else if ($\xi < P_4$)
  ...





- When sampling $\mathbf{x}_i$, use Russian roulette with $p_1(\mathbf{x}_i) = P_1(\mathbf{x}_i) = e^{-\sigma_{tr}\|\mathbf{x}_o - \mathbf{x}_i\|}$ to accept or reject a sample.

Progressive rendering of subsurface scattering

- The equation for reflected radiance:

$$L_r(\mathbf{x}_o, \vec{\omega}_o) = \int_A \int_{2\pi} S(\mathbf{x}_i, \vec{\omega}_i; \mathbf{x}_o, \vec{\omega}_o) L_i(\mathbf{x}_i, \vec{\omega}_i) \cos\theta_i \, \mathrm{d}\omega_i \, \mathrm{d}A_i \ .$$

- Initialize a frame by storing samples of transmitted light.
- For each sample:
  - Sample a point ($\mathbf{x}_i$) on the surface of the scattering material:
    - Sample a random triangle in the mesh ($\text{pdf}(\triangle) = A_\triangle/A$, where $A_\triangle$ is triangle area and A is total surface area. Use binary search with the face area cdf).
    - Sample a random point on the triangle ($\text{pdf}(\mathbf{x}_{i,\triangle}) = 1/A_\triangle$, use Barycentric coordinates).
  - Sample incident light L_i by sampling $\vec{\omega}_i$ using a cosine-weighted hemisphere ($\text{pdf}(\vec{\omega}_i) = \cos\theta_i/\pi$), for example.
  - Use $\vec{\omega}_i$ to find the direction of the transmitted ray and the Fresnel transmittance T_{12}.
  - Compute the transmitted radiance: $L_t = T_{12}L_i\cos\theta/\text{probabilities} = T_{12}L_i\pi A$.

Progressive rendering of subsurface scattering

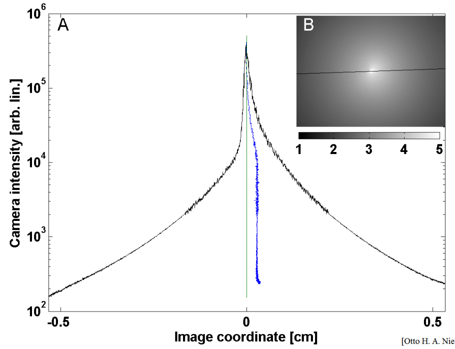
- The equation for reflected radiance:

$$L_r(\mathbf{x}_o, \vec{\omega}_o) = \int_A \int_{2\pi} S(\mathbf{x}_i, \vec{\omega}_i; \mathbf{x}_o, \vec{\omega}_o) L_i(\mathbf{x}_i, \vec{\omega}_i) \cos\theta_i \, \mathrm{d}\omega_i \, \mathrm{d}A_i \ .$$

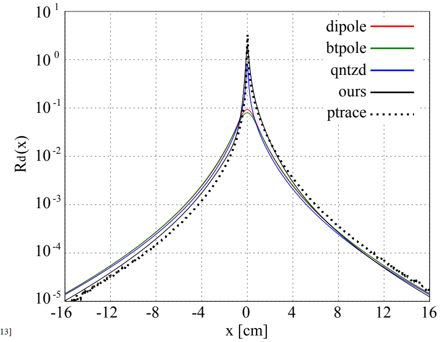
- To shade a ray hitting $\mathbf{x}_o$ with direction $-\vec{\omega}_o$:
  - Compute Fresnel transmittance T_{21} of the ray refracting from inside to $\vec{\omega}_o$.
  - Accept samples according to a Russian roulette using exponential distance attenuation as probability (rejection control).
  - Use L_t of accepted samples and T_{21} together with the analytical expression for S to Monte Carlo integrate the rendering equation.
  - The Monte Carlo estimator for the diffusive part is:

$$\begin{aligned} L_{d,N,M}(\mathbf{x}_o, \vec{\omega}_o) &= \frac{1}{NM}\sum_{p=1}^{M}\sum_{q=1}^{N} \frac{S(\mathbf{x}_{i,p}, \vec{\omega}_{i,q}; \mathbf{x}_o, \vec{\omega}_o) L_i(\mathbf{x}_{i,p}, \vec{\omega}_{i,q}) \cos\theta_i}{\text{pdf}(\mathbf{x}_{i,p})\text{pdf}(\vec{\omega}_{i,q})} \\ &= \frac{1}{NM}\sum_{p=1}^{M}\sum_{q=1}^{N} T_{21}\, S_d(\mathbf{x}_{i,p}, \vec{\omega}_{i,q}; \mathbf{x}_o, \vec{\omega}_o) L_t(\mathbf{x}_{i,p}, \vec{\omega}_{i,q}) \ . \end{aligned}$$

Profiles (diffuse reflectance curves)



[Otto H. A. Nielsen 2013]



- The directional model comes closer than the diffuse analytical models to measured and simulated diffuse reflectance curves.