Divide-and-Conquer

Inge Li Gørtz

Thank you to Kevin Wayne for inspiration to slides

Divide-and-Conquer

- Divide -and-Conquer.
  - Break up problem into several parts.
  - Solve each part recursively.
  - Combine solutions to subproblems into overall solution.
- Today
  - Mergesort (recap)
  - Recurrence relations
  - Integer multiplication

Mergesort

Recurrence relations

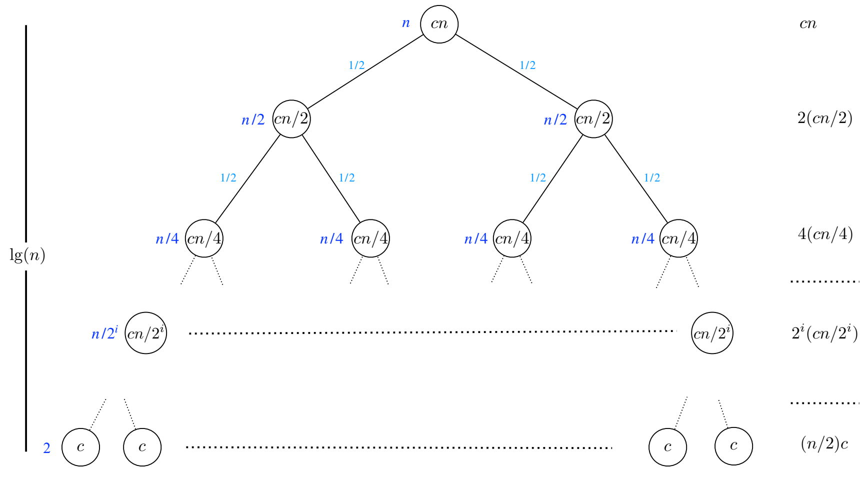
- T(n) = running time of mergesort on input of size n
- Mergesort recurrence:

$$T(n) \leq \begin{cases} 2T(n/2) + cn & \text{if } n > 2 \\ c & \text{otherwise} \end{cases}$$

- Solving the recurrence:
  - Recursion tree
  - Substitution

Mergesort recurrence: recursion tree

$$T(n) \le \begin{cases} 2T(n/2) + cn & \text{if } n > 2 \\ c & \text{otherwise} \end{cases}$$



Counting Inversions

Mergesort recurrence: substitution

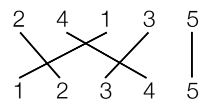
$$T(n) \le \begin{cases} 2T(n/2) + cn & \text{if } n > 2 \\ c & \text{otherwise} \end{cases}$$

- Substitute $T(n)$ with $kn \lg n$ and use induction to prove $T(n) \le n \lg nk$.
- Base case ($n = 2$):
  - By definition $T(2) = c$.
  - Substitution: $k \cdot 2 \lg 2 = 2k \ge c = T(2)$ if $k \ge c/2$.
- Induction: Assume $T(m) \le km \lg m$ for $m < n$.

$$\begin{aligned} T(n) &\le 2T(n/2) + cn \\ &\le 2k(n/2)\lg(n/2) + cn \\ &= kn(\lg n - 1) + cn \\ &= kn \lg n - kn + cn \\ &\le kn \lg n \quad \text{if} \quad k \ge c. \end{aligned}$$

Counting Inversions

- Given sequence (permutation) $a_1, a_2, \ldots, a_n$ of the numbers from 1 to n.
- Inversion: a_i and a_j inverted if $i < j$ and $a_i > a_j$.



- Applications:
  - Comparing preferences (e.g. on a music site).
  - Voting theory
  - Collaborative filtering
  - Measuring the “sortedness” of an array.
  - Sensitivity of Google’s ranking function.

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- Brute-force:
  - Compare each a_i with each a_j, where $i < j$.

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- Brute-force:
  - Compare each a_i with each a_j, where $i < j$.
  - Time: $O(n^2)$

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  - Divide: Split list in two.

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- Divide-and-Conquer:
  - Divide: Split list in two.
  - Conquer: recursively count inversions in each half.
  - Combine:
    - count inversions where a_i and a_j are in different halves
    - return sum.

`17 + 18 + 30 = 65`

Counting Inversions

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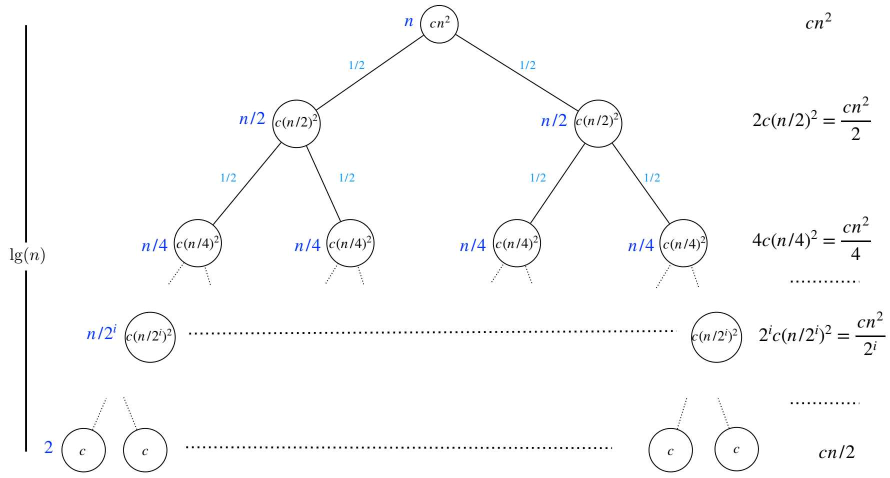
- Divide-and-Conquer:
  - Divide: Split list in two. *Divide: O(1)*
  - Conquer: recursively count inversions in each half. *Conquer: 2T(n/2)*
  - Combine: *Combine: ???*
    - count inversions where a_i and a_j are in different halves
    - return sum.

`17 + 18 + 30 = 65`

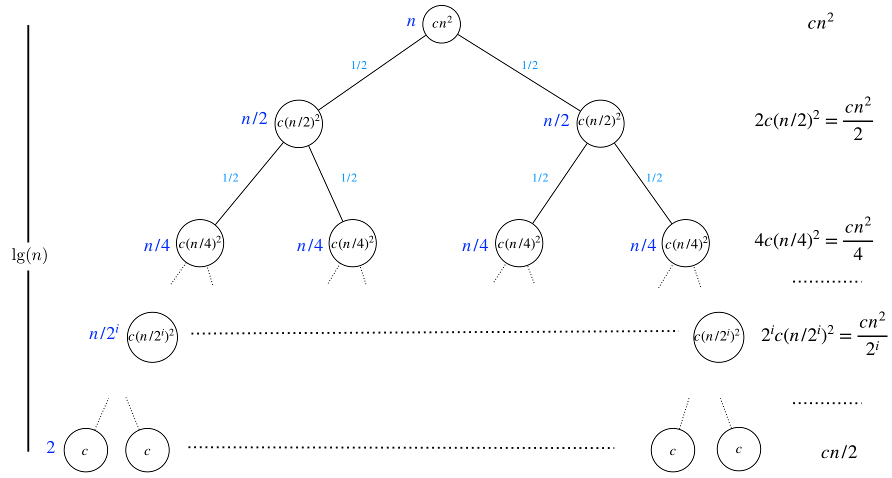
Another recurrence

$$T(n) \le \begin{cases} 2T(n/2) + cn^2 & \texttt{if } n > 2 \\ c & \texttt{otherwise} \end{cases}$$



More recurrences

$$T(n) \le \begin{cases} 2T(n/2) + cn^2 & \texttt{if } n > 2 \\ c & \texttt{otherwise} \end{cases} \qquad T(n) \le \sum_{i=0}^{\lg n} \frac{cn^2}{2^i} \le cn^2 \sum_{i=0}^{\lg n} \frac{1}{2^i} \le 2cn^2$$



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Counting Inversions: Combine

- Combine: count inversions where a_i and a_j are in different halves.

10	8	14	4

15	1	9	5

Counting Inversions: Combine

- Combine: count inversions where a_i and a_j are in different halves.
  - Assume each half sorted.

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Counting Inversions: Combine

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4	8	10	14		1	5	9	15

Counting Inversions: Combine

- Combine: count inversions where a_i and a_j are in different halves.
  - Assume each half sorted.

4	8	10	14		1	5	9	15
					4	3	2	0

Counting Inversions: Combine

- Combine: count inversions where a_i and a_j are in different halves.
  - Assume each half sorted.
  - Merge sorted halves into sorted whole while counting inversions.

4	8	10	14		1	5	9	15
					4	3	2	0

1	4	5	8	9	10	14	15

Inversions: 4 + 3 + 2 + 0 = 9

Counting Inversions

- Given sequence (permutation) $a_1, a_2, \ldots, a_n$ of the numbers from 1 to n.
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  - Combine: *Combine: O(n)*
    - Merge-and-Count.

Counting Inversions: Implementation

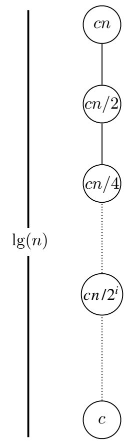
```
Sort-and-Count(L):

  if list L has one element:
      return (0, L)

  divide the list L into two halves A and B
  (iA, A) = Sort-and-Count(A)
  (iB, B) = Sort-and-Count(B)
  (iL, L) = Merge-and-Count(A, B)

  i = iA + iB + iL

  return (i, L)
```

- Pre-condition (Merge-and-Count): A and B are sorted.
- Post-condition (Sort-and-Count, Merge-and-Count): L is sorted.

More Recurrence Relations

More recurrence relations: 1 subproblem

$$T(n) \leq \begin{cases} T(n/2) + cn & \text{if } n > 2 \\ c & \text{otherwise} \end{cases}$$



- Summing over all levels:

$$T(n) \leq \sum_{i=0}^{\lg n - 1} \frac{cn}{2^i} = cn \sum_{i=0}^{\lg n - 1} \frac{1}{2^i} \leq 2cn = O(n)$$

- Substitution: Guess $T(n) \leq kn$
  - Base case:

$$k \cdot 2 \geq c = T(2) \quad \text{if} \quad k \geq c/2.$$

  - Assume $T(m) \leq km$ for $m < n$.

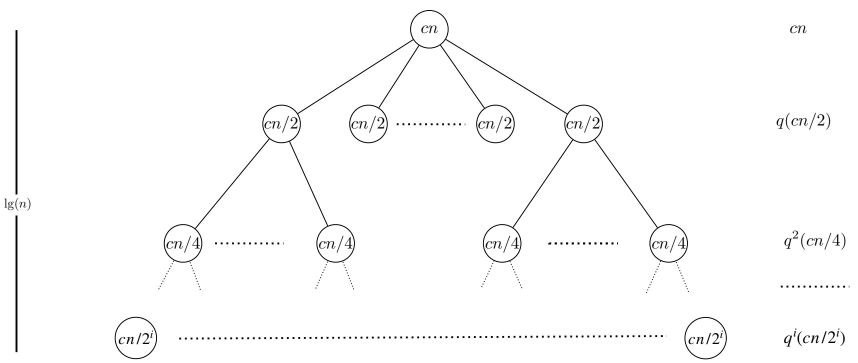
$$T(n) \leq T(n/2) + cn \leq k(n/2) + cn = (k/2)n + cn$$

$$\leq kn \quad \text{if} \quad c \leq k/2.$$

More than 2 subproblems

- q subproblems of size n/2.

$$T(n) \le \begin{cases} qT(n/2) + cn & \text{if } n > 2 \\ c & \text{otherwise} \end{cases}$$



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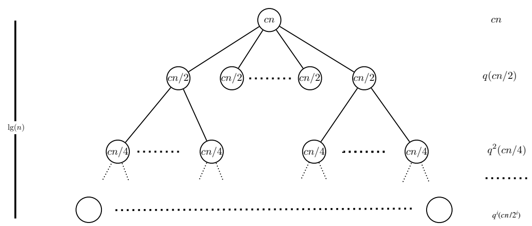
- Summing over all levels:

$$T(n) \le \sum_{j=0}^{\lg n - 1} \left(\frac{q}{2}\right)^j cn = cn \sum_{j=0}^{\lg n - 1} \left(\frac{q}{2}\right)^j$$



Geometric series.

for $x \neq 1: \sum_{i=0}^{m} x^i = \frac{x^{m+1} - 1}{x - 1}$

for $x < 1: \sum_{i=0}^{\infty} x^i = \frac{1}{1 - x}$

More than 2 subproblems

Proof of $cn \sum_{j=0}^{\lg n - 1} \left(\frac{q}{2}\right)^j = O(n^{\lg q})$

Geometric series.

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Use geometric series: $cn \sum_{j=0}^{\lg n - 1} \left(\frac{q}{2}\right)^j = cn \frac{\left(\frac{q}{2}\right)^{\lg n} - 1}{\frac{q}{2} - 1}$

Reduce $\left(\frac{q}{2}\right)^{\lg n} = \frac{q^{\lg n}}{2^{\lg n}} = \frac{q^{\lg n}}{n}$

Now:

$$cn \frac{\left(\frac{q}{2}\right)^{\lg n} - 1}{\frac{q}{2} - 1} = cn \frac{\frac{q^{\lg n}}{n} - 1}{\frac{q-2}{2}} = \frac{2c}{q-2} n \left(\frac{q^{\lg n}}{n} - 1\right) = \boxed{\frac{2c}{q-2}} (q^{\lg n} - n) = O(q^{\lg n})$$

constant

More than 2 subproblems

- q subproblems of size n/2.

$$T(n) \le \begin{cases} qT(n/2) + cn & \text{if } n > 2 \\ c & \text{otherwise} \end{cases}$$

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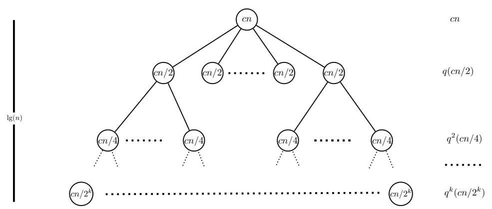

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Subproblems of different sizes

$$T(n) = \begin{cases} T(3n/4) + T(n/2) + f(n) & \text{if } n > 4 \\ c & \text{otherwise} \end{cases}$$