02941 Physically Based Rendering

Subsurface Scattering

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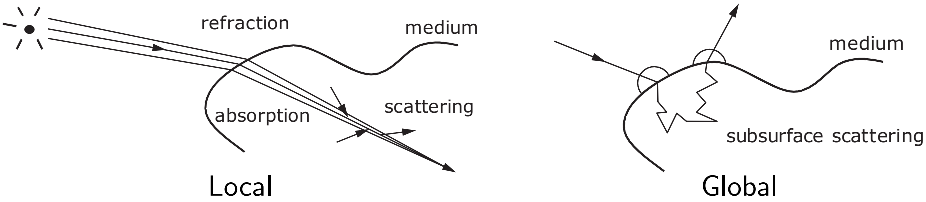
Volumes to surfaces

- From local to global formulation [Preisendorfer 1965].



- Suppose we let $L^0 = T_r(0, s)L(0)$ denote direct transmission.
- Preisendorfer introduces a scattering operator $\boldsymbol{S}_j$ denoting the illumination that has scattered j times inside the medium.
- The scattering operator provides a path from the radiative transfer equation (local) to the rendering equation (global):

$$L = L^0 \boldsymbol{S}_0 + \sum_{j=1}^{\infty} L^0 \boldsymbol{S}_j = \sum_{j=0}^{\infty} L^0 \boldsymbol{S}_j \, .$$

References

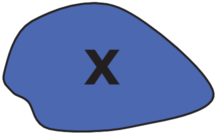
- Preisendorfer, R. W. *Radiative Transfer on Discrete Spaces*. Pergamon Press, 1965.

BSSRDF

- Since a monotone, bounded sequence of nonnegative real numbers $(\sum_{j=0}^{n} L^0 \boldsymbol{S}_j)$ converges to a real number (L), a global solution exists. Thus

$$L = \sum_{j=0}^{\infty} L^0 \boldsymbol{S}_j = L^0 \boldsymbol{S}\,.$$

- But what is $\boldsymbol{S}$?
- For



$$\rightarrow \bullet\, \boldsymbol{x} \quad , \quad \boldsymbol{S} \rightarrow \sigma_s p(\boldsymbol{x}, \vec{\omega}_i, \vec{\omega}_o)\,.$$

- Continuous boundary and interior leads from $\boldsymbol{S}$ to $S(X; \boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o)$, the so-called

  Bidirectional Scattering-Surface Reflectance Distribution Function.
- Originally called “the scattering function” [Venable and Hsia 1974]

  where it included time dependency and inelastic scattering. (Arguably a better name.)

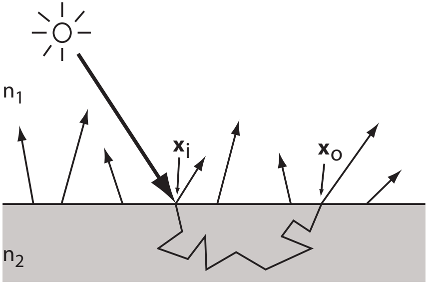
References

- Venable, Jr., W. H., and Hsia, J. J. Optical Radiation Measurement: Describing Spectrophotometric Measurements. Technical report, National Bureau of Standards (US), 1974.

Subsurface scattering

- Behind the rendering equation [Nicodemus et al. 1977]:

$$\frac{\mathrm{d}L_r(\boldsymbol{x}_o, \vec{\omega}_o)}{\mathrm{d}\Phi_i(\boldsymbol{x}_i, \vec{\omega}_i)} = S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o)$$



- An element of reflected radiance $\mathrm{d}L_r$ comes from an element of incident flux $\mathrm{d}\Phi_i$.
- S (the BSSRDF) is the proportionality factor between the two.
- Using the definition of radiance $L = \dfrac{\mathrm{d}^2\Phi}{\cos\theta\,\mathrm{d}A\,\mathrm{d}\omega}$, we have

$$L_r(\boldsymbol{x}_o, \vec{\omega}_o) = \int_A \int_{2\pi} S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) L_i(\boldsymbol{x}_i, \vec{\omega}_i) \cos\theta_i \,\mathrm{d}\omega_i \,\mathrm{d}A \,.$$

References

- Nicodemus, F. E., Richmond, J. C., Hsia, J. J., Ginsberg, I. W., and Limperis, T. Geometrical considerations and nomenclature for reflectance. Tech. rep., National Bureau of Standards (US), 1977.

An estimator for subsurface scattering

- The reflected radiance equation:

$$L_r(\boldsymbol{x}_o, \vec{\omega}_o) = \int_A \int_{2\pi} S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) L_i(\boldsymbol{x}_i, \vec{\omega}_i) \cos\theta_i \, \mathrm{d}\omega_i \, \mathrm{d}A_i \, .$$

- Monte Carlo estimator:

$$L_{r,N,M}(\boldsymbol{x}_o, \vec{\omega}_o) = \frac{1}{NM} \sum_{p=1}^{M} \sum_{q=1}^{N} \frac{S(\boldsymbol{x}_{i,p}, \vec{\omega}_{i,q}; \boldsymbol{x}_o, \vec{\omega}_o) L_i(\boldsymbol{x}_{i,p}, \vec{\omega}_{i,q}) \cos\theta_i}{\mathrm{pdf}(\boldsymbol{x}_{i,p}) \mathrm{pdf}(\vec{\omega}_{i,q})} \quad .$$

- Common direction sampling pdf (cosine-weighted hemisphere):

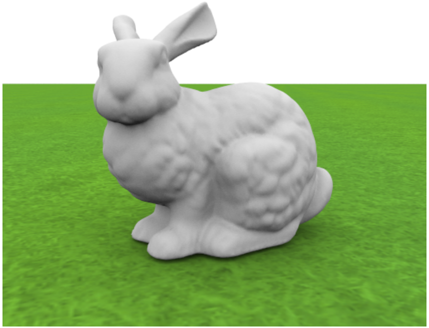
$$\mathrm{pdf}(\vec{\omega}_{i,q}) = \frac{\vec{\omega}_{i,q} \cdot \vec{n}_i}{\pi} = \frac{\cos\theta_i}{\pi} \quad .$$

- Common area sampling pdf (triangle mesh):

$$\mathrm{pdf}(\boldsymbol{x}_{i,p}) = \mathrm{pdf}(\triangle)\mathrm{pdf}(\boldsymbol{x}_{i,p,\triangle}) = \frac{A_\triangle}{A_\ell} \frac{1}{A_\triangle} = \frac{1}{A_\ell} \quad .$$

Sampling a cosine-weighted hemisphere (ambient occlusion)

- Material:
  $$S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) = \frac{\rho_d(\boldsymbol{x}_o)}{\pi}\delta(\boldsymbol{x}_o - \boldsymbol{x}_i)\,.$$
- Sampler:
  $$\mathrm{pdf}(\vec{\omega}_{i,q}) = \frac{\vec{\omega}_{i,q} \cdot \vec{n}_i}{\pi} = \frac{\cos\theta_i}{\pi}\,.$$
- Estimator:
  $$L_{r,N}(\boldsymbol{x}_o, \vec{\omega}_o)$$
  $$= \frac{1}{N}\sum_{q=1}^{N} \frac{\rho_d(\boldsymbol{x}_o)}{\pi} \frac{L_i(\boldsymbol{x}_o, \vec{\omega}_{i,q})\cos\theta_i}{\mathrm{pdf}(\vec{\omega}_{i,q})}$$
  $$= \rho_d(\boldsymbol{x}_o)\frac{1}{N}\sum_{q=1}^{M} L_i(\boldsymbol{x}_o, \vec{\omega}_{i,q})\,.$$



Sampling a triangle mesh (area lights, soft shadows)

- Material:

$$S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) = \frac{\rho_d(\boldsymbol{x}_o)}{\pi}\delta(\boldsymbol{x}_o - \boldsymbol{x}_i)\,.$$

- Sampler:

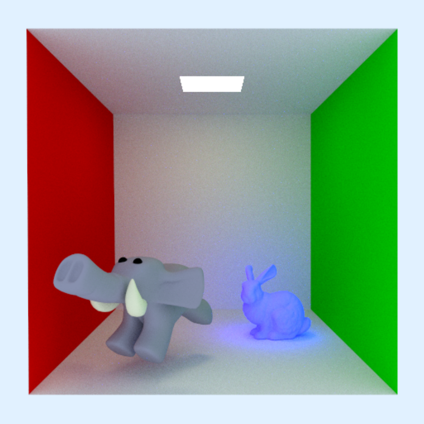
$$\vec{\omega}_{i,q} = \frac{\boldsymbol{x}_{\ell,q} - \boldsymbol{x}_i}{\|\boldsymbol{x}_{\ell,q} - \boldsymbol{x}_i\|}$$

$$\mathrm{pdf}(\boldsymbol{x}_{\ell,q}) = \mathrm{pdf}(\triangle)\mathrm{pdf}(\boldsymbol{x}_{\ell,q,\triangle}) = \frac{A_\triangle}{A_\ell}\frac{1}{A_\triangle}\,.$$

- Estimator:

$$L_{r,N}(\boldsymbol{x}_o, \vec{\omega}_o)$$

$$= \frac{\rho_d(\boldsymbol{x}_o)}{\pi}\frac{1}{N}\sum_{q=1}^{N} L_e(\boldsymbol{x}_{\ell,q}, -\vec{\omega}_{i,q})V(\boldsymbol{x}_{\ell,q}, \boldsymbol{x}_o)\frac{\cos\theta_i\cos\theta_\ell}{\|\boldsymbol{x}_{\ell,q} - \boldsymbol{x}_i\|^2}A_\ell\,.$$

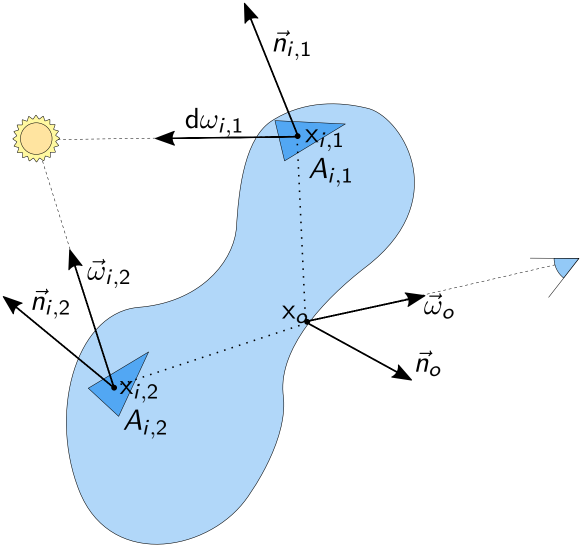


Sampling for subsurface scattering

- Material:
  $S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) = \ldots .$
- Sampler:

$$\mathrm{pdf}(\vec{\omega}_{i,q}) = \frac{\vec{\omega}_{i,q} \cdot \vec{n}_i}{\pi} = \frac{\cos\theta_i}{\pi} .$$

$$\mathrm{pdf}(\boldsymbol{x}_{i,p}) = \mathrm{pdf}(\triangle)\mathrm{pdf}(\boldsymbol{x}_{i,p,\triangle}) = \frac{A_\triangle}{A_\ell}\frac{1}{A_\triangle} = \frac{1}{A_\ell} .$$



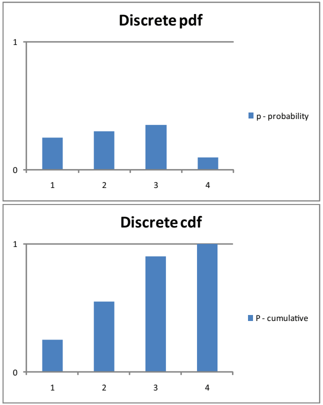
References

- Frisvad, J. R., Hachisuka, T., and Kjeldsen, T. K. Directional dipole model for subsurface scattering. *ACM Transactions on Graphics 34*(1), pp. 5:1–5:12, November 2014. Presented at SIGGRAPH 2015.
- Dal Corso, A., and Frisvad, J. R. Point cloud method for rendering BSSRDFs. Technical Report, Technical University of Denmark, 2018.

Rejection control

- The exponential attenuation $e^{-\sigma_{\mathrm{tr}} d}$ appears in all analytical BSSRDFs and $d \to \|\boldsymbol{x}_o - \boldsymbol{x}_i\|$ for $\|\boldsymbol{x}_o - \boldsymbol{x}_i\| \to \infty$.
- We should exploit this.
- Russian roulette:

  sample $\xi \in [0, 1]$ uniformly;
  if $(\xi < P_1)$
      call event 1;
      divide by p_1;
  else if $(\xi < P_2)$
      call event 2;
      divide by p_2;
  else if $(\xi < P_3)$
      ...
  else if $(\xi < P_4)$
      ...



- When sampling $\boldsymbol{x}_i$, use Russian roulette with $p_1(\boldsymbol{x}_i) = P_1(\boldsymbol{x}_i) = e^{-\sigma_{\mathrm{tr}}\|\boldsymbol{x}_o - \boldsymbol{x}_i\|}$ to accept or reject a sample.

Progressive rendering of subsurface scattering

- The equation for reflected radiance:

$$L_r(\boldsymbol{x}_o, \vec{\omega}_o) = \int_A \int_{2\pi} S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) L_i(\boldsymbol{x}_i, \vec{\omega}_i) \cos\theta_i \, \mathrm{d}\omega_i \, \mathrm{d}A_i \, .$$

- Initialize a frame by storing samples of transmitted light.
- For each sample:
  - Sample a point ($\boldsymbol{x}_{i,p}$) on the surface of the scattering material by sampling a random triangle and then a random point in the triangle: $\mathrm{pdf}(\boldsymbol{x}_{i,p}) = 1/A_\ell$.
  - Sample a ray direction $\vec{\omega}_{i,q}$ using a cosine-weighted hemisphere and trace they ray to collect incident light L_i: $\mathrm{pdf}(\vec{\omega}_{i,q}) = \cos\theta_i/\pi$.
  - Use $\vec{\omega}_{i,q}$ to find the direction of the transmitted/refracted ray and the Fresnel transmittance T_{12}.
  - Store the transmitted radiance: $L_t = \dfrac{T_{12} L_i \cos\theta_i}{\mathrm{pdf}(\boldsymbol{x}_{i,p})\mathrm{pdf}(\vec{\omega}_{i,q})} = T_{12} L_i \pi A_\ell$.

Progressive rendering of subsurface scattering

- The equation for reflected radiance:

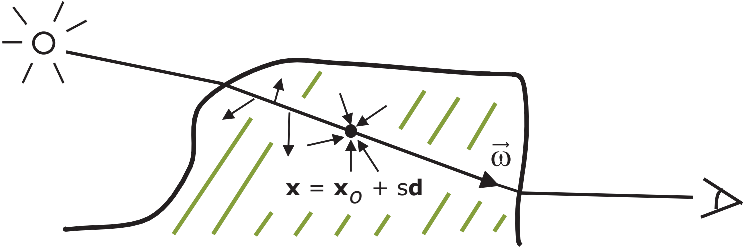
$$L_r(\boldsymbol{x}_o, \vec{\omega}_o) = \int_A \int_{2\pi} S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) L_i(\boldsymbol{x}_i, \vec{\omega}_i) \cos\theta_i \, \mathrm{d}\omega_i \, \mathrm{d}A_i \, .$$

- For a ray hitting $\boldsymbol{x}_o$ with direction $-\vec{\omega}_o$:
  - Compute Fresnel transmittance T_{21} of the ray refracting from inside to $\vec{\omega}_o$.
  - Loop through the NM samples using exponential distance attenuation as the probability of acceptance in a Russian roulette (rejection control).
  - Use L_t of accepted samples and T_{21} together with the analytical expression for S to Monte Carlo integrate the rendering equation.
  - The Monte Carlo estimator for the diffusive part is (we use $N = 1$):

$$\begin{aligned} L_{d,N,M}(\boldsymbol{x}_o, \vec{\omega}_o) &= \frac{1}{NM} \sum_{p=1}^{M} \sum_{q=1}^{N} \frac{S(\boldsymbol{x}_{i,p}, \vec{\omega}_{i,q}; \boldsymbol{x}_o, \vec{\omega}_o) L_i(\boldsymbol{x}_{i,p}, \vec{\omega}_{i,q}) \cos\theta_i}{\mathrm{pdf}(\boldsymbol{x}_{i,p}) \mathrm{pdf}(\vec{\omega}_{i,q})} \\ &= \frac{T_{21}}{NM} \sum_{p=1}^{M} \sum_{q=1}^{N} \frac{S_d(\boldsymbol{x}_{i,p}, \vec{\omega}_{i,q}; \boldsymbol{x}_o, \vec{\omega}_o) T_{12} L_i(\boldsymbol{x}_{i,p}, \vec{\omega}_{i,q}) \pi A_\ell}{e^{-\sigma_{\mathrm{tr}} \|\boldsymbol{x}_o - \boldsymbol{x}_{i,p}\|}} \left[\xi < e^{-\sigma_{\mathrm{tr}} \|\boldsymbol{x}_o - \boldsymbol{x}_{i,p}\|} \right] . \end{aligned}$$

Radiative transfer as diffusion

- The local formulation: Consider a point $\boldsymbol{x}$ along a ray traversing a medium in the direction $\vec{\omega}$. Then
$$(\vec{\omega} \cdot \nabla) L(\boldsymbol{x}, \vec{\omega}) = -\sigma_t(\boldsymbol{x}) L(\boldsymbol{x}, \vec{\omega}) + \sigma_s(\boldsymbol{x}) \int_{4\pi} p(\boldsymbol{x}, \vec{\omega}', \vec{\omega}) L(\boldsymbol{x}, \vec{\omega}')\, \mathrm{d}\omega' \, .$$

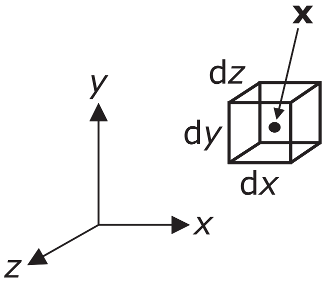


- We assume that the medium is turbid (scattering), but not emissive. (The emission term $L_e(\boldsymbol{x}, \vec{\omega})$ has been left out.)
- To find a global formulation, we think of multiple scattering as a diffusion process.

Fluence and vector irradiance

- In diffusion theory, we use quantities that describe the light field in an element of volume of the scattering medium.
- Total flux, or fluence, is defined by

$$\phi(\boldsymbol{x}) = \int_{4\pi} L(\boldsymbol{x}, \vec{\omega})\, \mathrm{d}\omega\,.$$



- Net flux, or vector irradiance, is

$$\boldsymbol{E}(\boldsymbol{x}) = \int_{4\pi} \vec{\omega} L(\boldsymbol{x}, \vec{\omega})\, \mathrm{d}\omega\,.$$

- $\boldsymbol{E}$ is measured in flux per (orthogonally) projected area.
- The areas are $\mathrm{d}y\,\mathrm{d}z$, $\mathrm{d}x\,\mathrm{d}z$, and $\mathrm{d}x\,\mathrm{d}y$.

Fick's law of diffusion

- The direction in which the fluence undergoes the greatest rate of increase is

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) .$$

- Intuitively it is then reasonable to assume the proportionality (*Fick's law of diffusion*)

$$\boldsymbol{E}(\boldsymbol{x}) = -D(\boldsymbol{x})\,\nabla\phi(\boldsymbol{x})\,.$$

- This assumption is only valid in the asymptotic regions of a medium, that is, in regions far enough from the boundaries to ensure that most light has suffered from multiple scattering events.
- The value of the diffusion coefficient D is also important for the correctness of the law.
- The standard value $D = 1/(3\sigma_t')$ is valid for nearly isotropic, almost non-absorbing materials.

The diffusion equation

- Integrating the radiative transfer equation over all outgoing directions results in

$$\int_{4\pi} (\vec{\omega} \cdot \nabla) L(\boldsymbol{x}, \vec{\omega}) \, \mathrm{d}\omega = - \int_{4\pi} \sigma_t(\boldsymbol{x}) L(\boldsymbol{x}, \vec{\omega}) \, \mathrm{d}\omega \; + \int_{4\pi} \sigma_s(\boldsymbol{x}) \int_{4\pi} p(\boldsymbol{x}, \vec{\omega}', \vec{\omega}) L(\boldsymbol{x}, \vec{\omega}') \, \mathrm{d}\omega' \, \mathrm{d}\omega \, .$$

- In terms of fluence ϕ and vector irradiance $\boldsymbol{E}$, this turns into

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x}) = -\sigma_t(\boldsymbol{x})\phi(\boldsymbol{x}) + \sigma_s(\boldsymbol{x})\phi(\boldsymbol{x}) = \; -\sigma_a(\boldsymbol{x})\phi(\boldsymbol{x}) \, .$$

- Inserting Fick's law ($\boldsymbol{E} = -D \, \nabla\phi$), we get the diffusion equation (for a non-emitter):

$$\nabla \cdot (D(\boldsymbol{x}) \, \nabla\phi(\boldsymbol{x})) = \sigma_a(\boldsymbol{x})\phi(\boldsymbol{x}) \, .$$

  which we can solve for ϕ to get the light field in a scattering medium.

Splitting up the BSSRDF

- Bidirectional scattering surface reflectance distribution function: $S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o)$.
- In asymptotic regions of the medium, we can use diffusion.
- Splitting up the BSSRDF

$$S(\boldsymbol{x}_i, \vec{\omega}_i; \boldsymbol{x}_o, \vec{\omega}_o) = T_o(\vec{\omega}_o, n_{\text{med}}, n_o) S_d(\|\boldsymbol{x}_o - \boldsymbol{x}_i\|) T_i(\vec{\omega}_i, n_i, n_{\text{med}}) \; + \; \boldsymbol{S}^1 \, .$$

  where
  - T_i and T_o are Fresnel transmittance terms.
  - $\boldsymbol{S}^1$ is the single scattering operator.
  - S_d is the diffusion part.
  - n_{med}, n_i, and n_o are the refractive indices of the scattering medium, the one from where light is incident (incoming), and the one from where light is emergent (outgoing).
- Note that the diffusion part depends only on the distance between the point of incidence ($\boldsymbol{x}_i$) and the point of emergence ($\boldsymbol{x}_o$).

The diffusion part of the BSSRDF

- Consider the diffusion part of the BSSRDF S_d only.

$$\frac{\mathrm{d}L_d(\boldsymbol{x}_o, \vec{\omega}_o)}{\mathrm{d}\Phi_i(\boldsymbol{x}_i, \vec{\omega}_i)} = S_d(\|\boldsymbol{x}_o - \boldsymbol{x}_i\|) \,.$$

- Radiance is $L = \dfrac{\mathrm{d}^2\Phi}{\cos\theta \,\mathrm{d}A \,\mathrm{d}\omega}$, radiant exitance is $M = \dfrac{d\Phi}{dA}$.
- By cosine-weighted integration over all outgoing directions, we have

$$\int_{2\pi} \frac{\mathrm{d}L_d(\boldsymbol{x}_o, \vec{\omega}_o)}{\mathrm{d}\Phi_i(\boldsymbol{x}_i, \vec{\omega}_i)} \cos\theta \,\mathrm{d}\omega = \int_{2\pi} S_d(\|\boldsymbol{x}_o - \boldsymbol{x}_i\|) \cos\theta \,\mathrm{d}\omega \,.$$

- Since S_d only depends on distance, we get

$$\frac{\mathrm{d}M_d(\boldsymbol{x}_o)}{\mathrm{d}\Phi_i(\boldsymbol{x}_i, \vec{\omega}_i)} = \pi S_d(\|\boldsymbol{x}_o - \boldsymbol{x}_i\|) \,.$$

Boundary conditions

- Assume that no diffuse radiance is scattered in the inward direction at the surface. Then

$$\left.\begin{aligned} \int_{-2\pi} L(\boldsymbol{x}, \vec{\omega}') \cos\theta \, \mathrm{d}\omega' &= 0 \\ \int_{2\pi} L(\boldsymbol{x}, \vec{\omega}') \cos\theta \, \mathrm{d}\omega' &= M_d(\boldsymbol{x}) \end{aligned}\right\} \Rightarrow M_d(\boldsymbol{x}) = \int_{4\pi} L(\boldsymbol{x}, \vec{\omega}') \cos\theta \, \mathrm{d}\omega' = \vec{n} \cdot \boldsymbol{E}(\boldsymbol{x}).$$

- The assumption rarely holds, but this gives a link between surface and volume rendering (the local and the global formulation of radiative transfer):

$$M_d(\boldsymbol{x}) = \vec{n} \cdot \boldsymbol{E}(\boldsymbol{x}) \,,$$

  where $\vec{n}$ is the unit length surface normal.

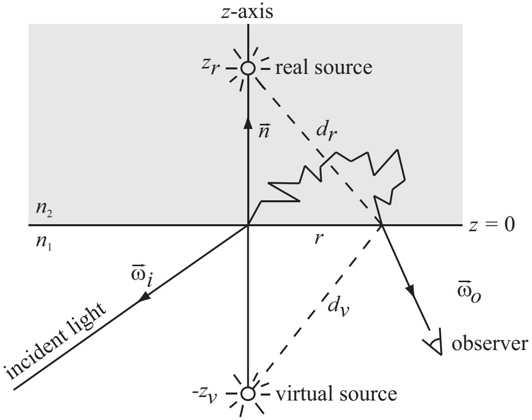
- This link is crucial for the model. A correction is used later to alleviate the error introduced by the incorrect assumption.

The dipole approximation

- An approximate solution for the diffusion equation:

$$\phi(z) = \frac{\Phi}{4\pi D}\left(\frac{e^{-\sigma_{\mathrm{tr}} d_r}}{d_r} - \frac{e^{-\sigma_{\mathrm{tr}} d_v}}{d_v}\right), \quad \sigma_{\mathrm{tr}} = \sqrt{3\sigma_a \sigma_t'}\,,\ \sigma_t' = \sigma_a + (1-g)\sigma_s\,.$$

- σ_s is the scattering coefficient.
- σ_a is the absorption coefficient.
- g is the asymmetry parameter.
- r is the distance between $\boldsymbol{x}_i$ and $\boldsymbol{x}_o$.



Putting the math together

- Fick's law ($\boldsymbol{E} = -D\,\nabla\phi$) and the boundary conditions:

$$M_d(\boldsymbol{x}) = \vec{n}\cdot\boldsymbol{E}(\boldsymbol{x}) = \,-\,D\,\vec{n}\cdot\,\nabla\phi(\boldsymbol{x})\,.$$

- Using the rendering equation $\dfrac{\mathrm{d}M_d}{\mathrm{d}\Phi_i} = \pi S_d$:

$$\pi S_d(\|\boldsymbol{x}_o - \boldsymbol{x}_i\|) = \,-\,D\frac{\mathrm{d}(\vec{n}\cdot\,\nabla\phi(\boldsymbol{x}_o))}{\mathrm{d}\Phi_i(\boldsymbol{x}_i,\vec{\omega}_i)}\,.$$

- Inserting the dipole approximation of ϕ:

$$\pi S_d(r) = -D\frac{\mathrm{d}\Phi}{\mathrm{d}\Phi_i}\frac{\partial}{\partial z}\left[\frac{1}{4\pi D}\left(\frac{e^{-\sigma_{\mathrm{tr}}d_r(z,r)}}{d_r(z,r)} - \frac{e^{-\sigma_{\mathrm{tr}}d_v(z,r)}}{d_v(z,r)}\right)\right]\,.$$

where

$$d_r(z,r) = \sqrt{r^2 + (z+z_r)^2} \quad \text{and} \quad d_v(z,r) = \sqrt{r^2 + (z-z_v)^2}\,.$$

The subsurface scattering model

- Taking the partial derivative with respect to z and afterwards setting $z = 0$:

$$S_d(r) = \frac{1}{4\pi^2}\frac{\mathrm{d}\Phi}{\mathrm{d}\Phi_i}\left(\frac{z_r(1+\sigma_{\mathrm{tr}}d_r)e^{-\sigma_{\mathrm{tr}}d_r}}{d_r^3} + \frac{z_v(1+\sigma_{\mathrm{tr}}d_v)e^{-\sigma_{\mathrm{tr}}d_v}}{d_v^3}\right).$$

  with $d_r(r) = \sqrt{r^2 + z_r^2}$ and $d_v(r) = \sqrt{r^2 + z_v^2}$.
- It remains now only to estimate Φ, z_r, and z_v.
- The model is based on *reduced* scattering properties:
  - Reduced scattering coefficient: $\sigma_s' = \sigma_s(1-g)$.
  - Reduced extinction coefficient: $\sigma_t' = \sigma_s' + \sigma_a$.
  - Reduced scattering albedo: $\alpha' = \sigma_s'/\sigma_t'$.
  - Transport mean free path: $\Lambda = 1/\sigma_t'$.
- From these we define: $\Phi = \alpha'\Phi_i$ (meaning $\frac{d\Phi}{d\Phi_i} = \alpha'$) and $z_r = 1/\sigma_t'$.
- The displacement of the virtual source z_v is corrected to mitigate the boundary condition error: $z_v = z_r + 4AD$, where $D = 1/(3\sigma_t')$ is the diffusion coefficient and A is the reflection (or the Groenhuis) parameter.

Validity

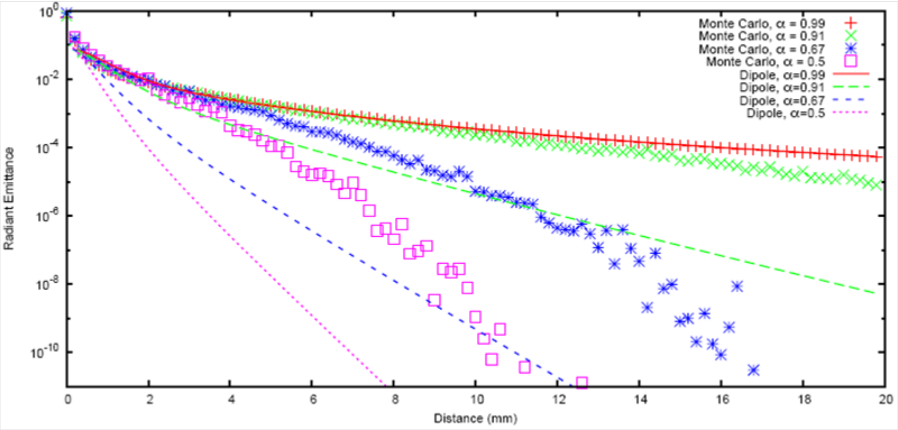


Figure from Craig Donner's PhD thesis [2006].

References

- Donner, C. *Towards Realistic Image Synthesis of Scattering Materials*, PhD thesis, University of California, San Diego, 2006.